A unique representation for the unitary time evolution operator of an N-mode quadratic Hamiltonian

Dharmik Patel

Abstract

We detail the method of representing the N-mode time evolution operator $\hat{U}_N(t)$ as a product of N-mode squeezing, displacement, and rotation operators up to a phase factor as outlined by X. Ma and W. Rhodes, hereafter referred to as 'the authors,' in [1]. The Wei-Norman Lie algebraic method of expressing the propagator for a system of first-order differential operator equations as a finite product of exponential operators is used in conjunction with disentangling and normal ordering techniques to construct and solve a system of differential equations arising from the time-dependent Schrödinger equation. The results are applied to a coupled channel waveguide - ring resonator system with an effective $\chi^{(2)}$ nonlinearity under the undepleted pump approximation and the resulting photon statistics are examined with respect to the input pulse energy U_P and duration τ_P.

Contents

1 Introduction and Lie algebraic preliminaries

1.1 The Lie algebra of constituent Hamiltonians

We consider Hamiltonians of the form

$$\hat{H}(t) = \sum_{i=1}^{n} f_i(t)\hat{H}_i \tag{1}$$

where $f_i(t)$ are a set of linearly independent and complex-valued functions of time, and $\hat{H}_i$ are constant Hamiltonian operators. The set of these operators,

$$\left\{\hat{H}_i \colon i = 1, 2, \ldots, n\right\} = \left\{\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n\right\} \tag{2}$$

forms a closed (under the Lie bracket[1]) n-dimensional[2] Lie algebra $\mathfrak{g}$. The set of n operators $\hat{H}_i$ in $\{\hat{H}_1, \ldots, \hat{H}_n\}$ (the basis of the Lie algebra $\mathfrak{g}$) can be enlarged to a Lie algebra $\mathcal{L}$ via repeated commutation, i.e. via linear combinations of possible Lie brackets of any two elements in $\mathfrak{g}$. Thus, we may consider $\{H_i\}$ to be the set of generators[3] of the Lie group $\mathcal{L}$, which is closed under the following commutation relation:

$$[\hat{H}_j, \hat{H}_k] = \sum_{l=1}^{n} \gamma_{jkl}\hat{H}_l \tag{3}$$

where γ_{ij} are the commutation structure constants [2]. The structure constants satisfy the following two conditions (the first arising from the skew symmetry of the Lie bracket, i.e. $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$ and the second arising from the Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$):

$$\begin{aligned} \gamma_{jkl} + \gamma_{kjl} &= 0 \\ \sum_n (\gamma_{jkn}\gamma_{nlm} + \gamma_{kln}\gamma_{njm} + \gamma_{ljn}\gamma_{nkm}) &= 0 \end{aligned}$$

[1] In general, there is no method to determine the time evolution operator of an arbitrary time-dependent Hamiltonian. However, due to the nature of the quadratic Hamiltonian, we can exploit the symmetries corresponding to the Lie group of the Lie algebra $\mathfrak{g}$ using the method of Wei and Norman.

[2] The Hamiltonian is most often rewritten to minimise the dimension of the associated Lie algebra, i.e. the basis of the Lie algebra is as small as possible yet contains all the terms of the Hamiltonian.

[3] This set 'generates' $\mathcal{L}$ in the sense that every element of $\mathcal{L}$ is expressible as a linear combination of Lie brackets of elements of $\mathfrak{g}$.

for all j, k, l, m.

1.2 Unitary evolution operator representations

The Hamiltonian operator governs the evolution of quantum states via the time-dependent Schrödinger equation:

$$i\hbar\partial_t\psi(t) = \hat{H}(t)\psi(t). \tag{4}$$

We may write ψ (which may be time-dependent) as

$$\psi(t) = \hat{U}(t)\psi_0 \tag{5}$$

where $\psi_0 = \psi(t_0)$ is the initial condition for (4) and $\hat{U}(t)$ is the time-evolution operator. Hence, we have a first-order linear differential equation for $\hat{U}(t)$:

$$i\hbar\frac{\partial}{\partial t}\hat{U}(t) = \hat{H}(t)\hat{U}(t), \quad \hat{U}(0) = I \tag{6}$$

where I is the identity operator. In [3], J. Wei and E. Norman showed that the time-evolution operator for Hamiltonians of the form equation (1) can be represented as

$$\hat{U}(t) = \prod_{i=1}^{n} \exp\Big[c_i(t)\hat{H}_i\Big] \tag{7}$$

where $c_i(t)$ are complex-valued scalar functions of time. Note the key difference between this representation of the time evolution operator and the result of W. Magnus [4]; [4] the expression of the time evolution operator as the exponential of a sum is valid only in a neighbourhood of the origin. On the other hand, the representation of the solution as a product of exponential factors is global for all solvable[5] Lie algebras [2]. Such a representation is known [5] for the time-evolution operator generated by the Hamiltonian that describes a single-mode Gaussian squeezed state of the quantum harmonic oscillator via the product of the squeeze, displacement, and rotation exponential operators:

$$\hat{H}(t) = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2(t)[\hat{q} + d(t)]^2 \qquad (\hbar \equiv 1) \tag{8}$$

where $\hat{p}$ and $\hat{q}$ are the familiar generalised position coordinates and momenta, $\omega(t)$ is the oscillation frequency and $d(t)$ is the displacement in position with initial condition $d(0) = 0$ due to the effect of a variable external force that comes into play at $t = 0$.

$$\hat{U}(t) = \exp[i\gamma(t)]\hat{S}(z(t))\hat{D}(\alpha(t))\hat{R}(\phi(t)) \tag{9}$$

where

$$\hat{S}(z) \equiv \exp\left(\frac{z(\hat{a}^\dagger)^2}{2} - \frac{z^*\hat{a}^2}{2}\right) \tag{10}$$

$$\hat{D}(\alpha) \equiv \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right) \tag{11}$$

$$\hat{R}(\phi) \equiv \exp\left(i\phi\hat{a}^\dagger\hat{a}\right). \tag{12}$$

We want to show that such a representation is achievable for N-mode quadratic Hamiltonians using the N-mode counterparts of the squeeze, displacement, and rotation exponential operators.

---

[4] If $\{\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_n\}$ is a basis for $\mathfrak{g}$, then the solution of the first order equation can be expressed as $\hat{U}(t) = \exp\left(\sum_{i=1}^{n} d_i(t)\hat{H}_i\right)$ where $d_i(t)$ are $\mathbb{C}-$valued functions of time.

[5] The set of elements of the Lie algebra that arise from the commutation of two 'pure' Lie elements is the 1st derived algebra. The 2nd derived algebra is defined as the derived algebra of the 1st derived algebra, and so on. A solvable Lie algebra is defined by $L^{(h)} = \{0\}$ where the superscript h indicates the h-th derived algebra.

1.3 Definitions of the N-mode squeezing, displacement, rotation, and ladder operators

First, we define the N-mode ladder (i.e. annihilation and creation) operators:

$$\hat{a} \equiv \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_N \end{bmatrix} \tag{13}$$

$$\hat{a}^\dagger \equiv \begin{bmatrix} \hat{a}_1^\dagger \\ \vdots \\ \hat{a}_N^\dagger \end{bmatrix} \tag{14}$$

where $\hat{a}_i$ and $\hat{a}_i^\dagger$ represent the annihilation and creation operators for the ith mode respectively. The N-mode squeeze, displacement, and rotation operators are defined as follows:

$$\hat{S}_N(z) \equiv \exp\left[\frac{(\hat{a}^\dagger)^\intercal z \hat{a}^\dagger}{2} - \frac{\hat{a}^\intercal z^\dagger \hat{a}}{2}\right] \tag{15}$$

$$\hat{D}_N(\alpha) \equiv \exp\left[\alpha^\intercal \hat{a}^\dagger - \alpha^\dagger \hat{a}\right] \tag{16}$$

$$\hat{R}_N(\Phi) \equiv \exp\left[i(\hat{a}^\dagger)^\intercal \Phi \hat{a}\right] \tag{17}$$

where z is an $N \times N$ matrix[6] defined as $z = re^{i\theta} = e^{i\theta^\intercal} r^\intercal$, where r is the squeeze parameter ($0 \leqslant [r]_{ij} < \infty$) and θ is the squeezing angle ($0 \leqslant [\theta]_{ij} \leqslant 2\pi$). Both r and θ are Hermitian matrices and additionally r is positive semidefinite. $\Phi = \Phi^\dagger$ is an $N \times N$ Hermitian matrix, α is the amount of displacement in optical phase space and $\alpha^\dagger \equiv \left[\alpha_1^*, \dots, \alpha_N^*\right]$. Note that each of the operators $\hat{S}_N(z), \hat{R}_N(\Phi)$, and $\hat{D}_N(\alpha)$ are unitary by definition.

2 Disentangling $\hat{S}_N$

2.1 Outline of the disentangling process

The N-mode squeeze operator is disentangled using the Baker-Campbell-Hausdorff (BCH) relation, which is derived by Lie algebra matrix techniques. Essentially, by defining operators that form an infinite dimensional Lie algebra and examining their commutators with each other, the authors see that the Lie algebra is homomorphic to the Lie algebra su$(1,1)$ i.e. there exists a commutator-preserving map between the infinite dimensional Lie algebra formed by their defined operator set and the Lie algebra underlying the special unitary group $\mathrm{SU}(1,1) \equiv \left\{\begin{bmatrix} u & v \\ v^* & u^* \end{bmatrix} | u, v \in \mathbb{C}, \quad uu^* - vv^* = 1\right\}$. The action of this homomorphism on the defined operators form a new set of algebraic elements that satisfy some given commutation relations. This allows for a factorisation[7] of the matrix $\exp\left[A^\dagger(z) - A(z)\right]$ as $\exp\left[A^\dagger(T)\right] \exp\left[B(\ln S)\right] \exp\left[-A(T)\right]$ where $z = re^{i\theta}$ was our symmetric $N \times N$ matrix, $T \equiv \tanh(r)e^{i\theta}$, and $S \equiv \operatorname{sech}(r)$. As the disentangled form of the N-mode squeezing operator is uniquely determined by the aforementioned infinite dimensional Lie algebra formed by the defined operator set, this factorisation yields the following disentangled form of $\hat{S}_N(z)$:

$$\hat{S}_N(z) = |S|^{\frac{1}{2}} \exp\left[\frac{1}{2}(\hat{a}^\dagger)^\intercal T \hat{a}^\dagger\right] \exp\left[(\hat{a}^\dagger)^\intercal (\ln S)\hat{a}\right] \exp\left[-\frac{1}{2}\hat{a}^\intercal T^\dagger \hat{a}\right] \tag{18}$$

[6]Assumed to be symmetric for convenience, as the authors do.
[7]Refer to Appendix A for the proof.

where $|S|^{\frac{1}{2}} \equiv \exp\left[\mathrm{Tr}(\ln S)\right]$. Furthermore, the authors put $\hat{S}_N(z)$ into normal ordered form by proving the following general result for an arbitrary matrix M:

$$\exp\left[(\hat{a}^\dagger)^\intercal M \hat{a}\right] = \sum_{n=0}^{\infty} \frac{:[(\hat{a}^\dagger)^\intercal (e^M - I)\hat{a}]^n:}{n!} = \sum_{n=0}^{\infty} \sum_{\{n_{ij}\}} \prod_{i,j=1}^{N} \frac{(e^M - I)_{ij}^{n_{ij}}}{n_{ij}!} \prod_{k=1}^{N} \hat{a}_k^{\dagger q_k} \prod_{k=1}^{N} \hat{a}_k^{p_k} \tag{19}$$

where $q_k = \sum_m n_{km}$ and $p_k = \sum_m n_{mk}$. The notation $:\ :$ denotes normal ordering without making use of the standard commutation relation (defined in Appendix B), and $\sum_{\{n_{ij}\}}$ indicates summation over all partitions of $n = \sum_{i,j=1}^{N} n_{ij}$ which are defined to be the individual sums of positive integers to the integer n. The proof is shown in Appendix B. Applying the theorem to $\exp\left[(\hat{a}^\dagger)^\intercal (\ln S)\hat{a}\right]$ for $M = \ln S$ [8], we derive

$$\hat{S}_N(z) = |S|^{\frac{1}{2}} \exp\left[\frac{1}{2}(\hat{a}^\dagger)^\intercal T \hat{a}^\dagger\right] \left[\sum_{n=0}^{\infty} \frac{:[(\hat{a}^\dagger)^\intercal (S - I)\hat{a}]^n:}{n!}\right] \exp\left[-\frac{1}{2}\hat{a}^\intercal T^\dagger \hat{a}\right]. \tag{20}$$

The general theorem of equation (19) is also used to derive the normal ordered form for the operator product $\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi)$:

$$\begin{aligned}\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) = {} & |S|^{\frac{1}{2}} \exp\left[-\frac{1}{2}(\alpha^\dagger\alpha + \alpha^\intercal T^\dagger \alpha)\right] \exp\left[\alpha^\intercal S^\intercal \hat{a}^\dagger + \frac{1}{2}(\hat{a}^\dagger)^\intercal T \hat{a}^\dagger\right] \\ & \times \left[\sum_{n=0}^{\infty} \frac{:[(\hat{a}^\dagger)^\intercal (S e^{i\Phi} - I)\hat{a}]^n:}{n!}\right] \exp\left[-(\alpha^\intercal T^\dagger + \alpha^\dagger) e^{i\Phi}\hat{a} - \frac{1}{2}\hat{a}^\intercal e^{i\Phi^\intercal}\hat{a}\right].\end{aligned} \tag{21}$$

This derivation is shown in Appendix B. This normal ordering process will prove to be almost parallel to the process used to put the N-mode time-evolution operator from the exponential factor form into the normal ordered form.

2.2 The general N-mode quadratic Hamiltonian and the expected form of the evolution operator

First, we introduce the N-mode quadratic Hamiltonian in its general form as presented by the authors:

$$\hat{H}_N(t) = (\hat{a}^\dagger)^\intercal \omega(t)\hat{a} + (\hat{a}^\dagger)^\intercal f(t)\hat{a}^\dagger + (\hat{a})^\intercal f^\dagger(t)\hat{a} + g^\intercal(t)\hat{a}^\dagger + g^\dagger(t)\hat{a} + h(t) \tag{22}$$

where $\omega(t)$ is an $N \times N$ Hermitian matrix, $f(t)$ is an $N \times N$ symmetric matrix, $g(t)$ is an $N \times 1$ matrix, and $h(t)$ is a real function. We want to show that the representation for the time-evolution operator of the N-mode quadratic Hamiltonian can be put into a form similar to equation (9):

$$\hat{U}_N(t) = \exp\left[i\gamma_N(t)\right] \hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) \tag{23}$$

The key difference between the operator product in equation (21) and the exponential factor product form in equation (9) is the new factor, an overall phase factor $\exp\left[i\gamma_N(t)\right]$. Using normal ordering techniques (as shown in Appendix B), we can express the time-evolution operator as

$$\hat{U}_N(t) = \exp\left[A(t)\right] \exp\left[B^\intercal(t)\hat{a}^\dagger + (\hat{a}^\dagger)^\intercal C(t)\hat{a}^\dagger\right] \left[\sum_{n=0}^{\infty} \frac{:[(\hat{a}^\dagger)^\intercal D(t)\hat{a}]^n:}{n!}\right] \exp\left[E^\intercal(t)\hat{a} + \hat{a}^\intercal F(t)\hat{a}\right] \tag{24}$$

[8] What does the logarithm of an operator mean? It is equivalent to talking about the logarithm of a matrix. (Hall 2015) defines it as

$$\log A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - I)^m}{m}$$

in Theorem 2.8. It is well-defined and continuous on the set of all $n \times n$ matrices A with $\|A - I\| < 1$

where $A(t), B(t), C(t), D(t), E(t)$, and $F(t)$ are functions of time with the initial condition

$$A(0) = B(0) = C(0) = D(0) = E(0) = F(0) = 0 \tag{25}$$

as the initial condition for the time-evolution operator is $\hat{U}_N(t = 0) = I$, the identity. The normal ordered form of the time-evolution operator is still written as the product of exponential operators, but we introduce the functions of time $A(t), \dots, F(t)$ that are coefficients of the basis elements of the Lie algebra generated by the N-mode quadratic Hamiltonian. Note that there are six such functions because there are six different yet equivalent ways to order[9] the product of the exponentials of the generators (as written in (7)), each with its own set of coefficients. We want to find these functions to determine the normal ordered form of the N-mode time-evolution operator and in doing so, prove that we may write the operator as equation (22) with the appropriate phase factor in terms of these functions.

3 Solving for the parameter functions

To do this, we examine the Schrödinger equation for the N-mode time-evolution operator. Expanding $\hat{H}_N(t)\hat{U}_N(t)$ on the right hand side of the Schrödinger equation:

$$i\hbar\frac{\partial U_N(t)}{\partial t} = \hat{H}_N(t)\hat{U}_N(t) \tag{26}$$

and comparing terms with identical factors composed of a combination of the four operators $\hat{a}^\dagger, \hat{a}, (\hat{a}^\dagger)^\intercal, (\hat{a})^\intercal$ on the left hand side of (26) (after simplification and grouping terms with like operator factors) we see that these factors comprised of the four aforementioned operators are the basis elements of the Lie algebra generated by our Hamiltonian $\hat{H}_N(t)$, and comparing the coefficients of the like terms, we derive the following six differential equations[10]:

$$i\frac{\partial}{\partial t}A(t) = \mathrm{Tr}\left[f^\dagger(2C(t) + B(t)B^\intercal(t)) + g^\dagger(t)B(t) + h(t)\right], \tag{27}$$

$$i\frac{\partial}{\partial t}B(t) = (4C(t)f^\dagger(t) + \omega(t))B(t) + 2C(t)g^*(t) + g(t), \tag{28}$$

$$i\frac{\partial}{\partial t}C(t) = 4C(t)f^\dagger(t)C(t) + 2\omega(t)C(t) + f(t), \tag{29}$$

$$i\frac{\partial}{\partial t}D(t) = (4C(t)f^\dagger(t) + \omega(t))(D(t) + I), \tag{30}$$

$$i\frac{\partial}{\partial t}E(t) = (D^\intercal(t) + I)(2f^\dagger(t)B(t) + g^*(t)), \tag{31}$$

$$i\frac{\partial}{\partial t}f(t) = (D^\intercal(t) + I)f^\dagger(t)(D(t) + I). \tag{32}$$

Note that $\hat{U}_N(t)$ is unitary, i.e.

$$\hat{U}_N\hat{U}_N^* = \hat{U}_N^*\hat{U}_N = I \tag{33}$$

and the following commutators (proven explicitly in Appendix C):

$$\left[\hat{a}, \hat{U}_N(t)\right] = \hat{a}\hat{U}_N(t) - \hat{U}_N(t)\hat{a} = \frac{\partial}{\partial \hat{a}^\dagger}\hat{U}_N(t), \tag{34}$$

$$\left[\hat{a}^\dagger, \hat{U}_N(t)\right] = \hat{a}^\dagger\hat{U}_N(t) - \hat{U}_N(t)\hat{a}^\dagger = -\frac{\partial}{\partial \hat{a}}\hat{U}_N(t). \tag{35}$$

By the unitarity of the time evolution operator, we see that $E(t)$ and $F(t)$ are related to $B(t), C(t)$ and $D(t)$ since the differential equations[11] for E and F involve D and B, and those

---

[9] Simply because $3! = 6$.

[10] See Appendix D.

[11] We are dropping the argument "(t)" from the following sections for cleanliness.

for B, C, D involve only each other (save for C, which involves only itself). From the normal ordered form of the time evolution operator and the commutation relations shown above[12], we can write the transformation of the $\hat{a}$ and $\hat{a}^\dagger$ operators via the time evolution operator, i.e.

$$\hat{U}_N^\dagger(t)\hat{a}\hat{U}_N(t) = (I - 4CC^\dagger)^{-1}\left[(D+I)\hat{a} + 2C((D^\intercal)^\dagger + 1)\hat{a}^\dagger + (2CB^* + B)\right], \tag{36}$$

$$\hat{U}_N^\dagger(t)\hat{a}^\dagger\hat{U}_N(t) = (D^\intercal + I)^{-1}(-2F\hat{a} + \hat{a}^\dagger - E). \tag{37}$$

The Hermitian conjugate of the first transformation equation is

$$\begin{aligned}\left(\hat{U}_N^\dagger(t)\hat{a}\hat{U}_N(t)\right)^* &= \left((I - 4CC^\dagger)^{-1}\left[(D+I)\hat{a} + 2C((D^\intercal)^\dagger + 1)\hat{a}^\dagger + (2CB^* + B)\right]\right)^* \\ &= \left[(D+I)\hat{a} + 2C((D^\intercal)^\dagger + 1)\hat{a}^\dagger + (2CB^* + B)\right]^*\left((I - 4CC^\dagger)^{-1}\right)^*. \end{aligned} \tag{38}$$

Simplifying,

$$\begin{aligned}&\left[((D+I)\hat{a} + 2C((D^\intercal)^\dagger + 1)\hat{a}^\dagger)^* + (2CB^* + B)^*\right]\left((I^* - (4CC^\dagger)^*)^{-1}\right) \\ &\quad = \left[D^*\hat{a} + I\hat{a} + 2((D^\intercal)^\dagger + 1)^* C^*\hat{a}^\dagger + 2BC^* + B^*\right]\left((I^* - 4(C^\dagger)^* C^*)^{-1}\right). \end{aligned} \tag{39}$$

Comparing this with the second transformation equation, we find that

$$F = -(D^* + I)^{-1}C^\dagger(D+I) \tag{40}$$

$$E = -(D^* + I)^{-1}(2C^\dagger B + B^*) \tag{41}$$

$$I - 4CC^\dagger = (D+I)(D^\dagger + I). \tag{42}$$

This last equation immediately implies $I - 4CC^\dagger$ is non-negative. So, defining

$$C \equiv \frac{1}{2}\tanh(r)e^{i\theta} \tag{43}$$

where the squeezing parameter r is non-negative, and

$$B \equiv \operatorname{sech}(r)\alpha \tag{44}$$

we have

$$I - 4CC^\dagger = \operatorname{sech}^2(r) \tag{45}$$

and

$$D + I = \operatorname{sech}(r)e^{i\Phi}. \tag{46}$$

Then, the first transformation equation reads

$$\hat{U}_N^\dagger(t)\hat{a}\hat{U}_N(t) = \left[\cosh(r)e^{i\Phi}\hat{a} + \sinh(r)e^{i\theta}e^{-i\Phi^\intercal}\hat{a}^\dagger + \cosh(r)e^{i\Phi}\alpha + \sinh(r)e^{i\theta}e^{-i\Phi^\intercal}\alpha^*\right] \tag{47}$$

$$= \hat{R}_N^\dagger(\Phi)\hat{D}_N^\dagger(\alpha)\hat{S}_N^\dagger(z)\hat{a}\hat{S}_N^\dagger(z)\hat{D}_N^\dagger(\alpha)\hat{R}_N^\dagger(\Phi) \tag{48}$$

via the transformation equations

$$\hat{S}_N^\dagger(z)\hat{a}\hat{S}_N(z) = \cosh(r)\hat{a} + \sinh(r)e^{i\theta}\hat{a}^\dagger \tag{49}$$

$$\hat{D}_N^\dagger(\alpha)\hat{a}\hat{D}_N(\alpha) = \hat{a} + \alpha. \tag{50}$$

[12]The normal ordered form of the time evolution operator (23) and the commutation relations for $\hat{a}$ and $\hat{a}^\dagger$ with it can be used to

4 The phase factor and final operator product form of $\hat{U}_N$

Now, this form of the first transformation equation (48) implies that the time evolution operator can be written as (22), and if we compare (22) with the normal ordered form of the operator product

$$\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi)$$

which [1] states is

$$\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) = \sqrt{|S|}\exp\left[-\frac{1}{2}(\alpha^\dagger\alpha + \alpha^\intercal T^\dagger\alpha)\right]\exp\left[\alpha^\intercal S^\intercal\hat{a}^\dagger + \frac{1}{2}(\hat{a}^\dagger)^\intercal T\hat{a}^\dagger\right]$$
$$\times\left[\sum_{n=0}^{\infty}\frac{:[(\hat{a}^\dagger)^\intercal(Se^{i\Phi}-I)\hat{a}]^n:}{n!}\right]\exp\left[-(\alpha^\intercal T^\dagger + \alpha^\dagger)e^{i\Phi}\hat{a} - \frac{1}{2}(\hat{a})^\intercal e^{i\Phi^\intercal}T^\dagger e^{i\Phi}\hat{a}\right] \quad (51)$$

and use the fact that the differential equation for C is dependent on C itself, we can solve for the phase factor

$$\gamma_N = \operatorname{Im}\left(A + \alpha^\intercal C^\dagger\alpha\right). \quad (52)$$

4.1 Uniqueness of the operator product form

We know the expression

$$\hat{U}_N(t) = \exp\left[i\operatorname{Im}\left(A + \alpha^\intercal C^\dagger\alpha\right)\right]\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) \quad (53)$$

is unique as the differential equation for the parameter function C is actually a Riccati equation; a first order ordinary differential equation which is quadratic in C. It is not necessary for the other five equations to be of the Riccati form since all six equations are coupled; the presence of the Riccati form for the differential equation for C is enough to guarantee the uniqueness of our solution from the implications of the uniqueness and existence of solutions for the Riccati equation. The general matrix Riccati equation is written as follows:

$$XDX + XA + A^*X - W = 0$$

where $D \geqslant 0$ and $W^* = W$. Comparing this to our equation for C, we have

$$X = C, \quad D = f^\dagger, \quad A = \omega \quad \text{and} \quad W = f - i\dot{C}$$

and so the existence and uniqueness result for the Riccati equation can also be formulated for the equation for C. This equation can be solved (in general) by transforming it into a homogenous second-order ODE and then utilising a general solution to determine the necessary coefficients [6], [7].

5 Ring resonator system simulation results

The results we obtained can be applied to a coupled channel waveguide - ring resonator system with an effective $\chi^{(2)}$ nonlinearity confined to an interaction region limited to the ring. We do not provide a complete theoretical background here; we refer the reader to the list of references at the end of this section for a more complete treatment. We introduce the ring resonator system, the governing Hamiltonian, the system of differential equations describing the dynamics of the system (with respect to squeezed light generation) and the results of our simulations solving the system with respect to the following input pulse characteristics: pulse energy, pulse duration, and the real channel - ring coupling efficiency. Our simulations model lossy generation via the implementation of a phantom channel.

5.1 The ring resonator system

The coupled channel waveguide - ring resonator system is an example of an integrated photonic structure; the channel waveguide serves to confine and guide light towards the ring resonator which is a variety of waveguide that forms a loop, such that a resonance is observed when the path length of the loop equals some integral multiple of the wavelength of incoming light. A diagram is shown below in Figure 1. For an extensive review of ring resonator structures and parameters, see [8].

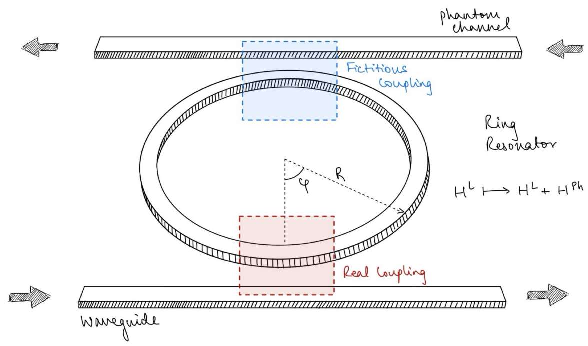


Figure 1: Diagram of a coupled channel waveguide - ring resonator with an additional fictitious coupled channel waveguide called the 'phantom channel'.

The process of generating squeezed light via the ring resonator is nonlinear with a $\chi^{(2)}$ nonlinearity. We utilise the formalism provided by asymptotic in and out states [9]. For background on a full treatment of nonlinear quantum optics (including discussions on generating squeezed light via ring resonators), we refer the reader to [10].

The Hamiltonian of the system is

$$H(t) = \hbar \sum_{k,l} \Delta_{kl}(t) \hat{a}_k^\dagger \hat{a}_l + \hbar \sum_{k,l} \zeta_{kl}(t) \hat{a}_k^\dagger \hat{a}_l^\dagger + \text{H.c.} \tag{54}$$

The Hamiltonian has the form $H = H^L + H^{NL}$ with H^L the linear contribution and H^{NL} the nonlinear contribution. In general, the Hamiltonian for such a system would not be quadratic; it would consist of higher order terms in $\hat{a}$ and $\hat{a}^\dagger$. The reason we are able to use the results we have obtained in the previous sections is the undepleted pump approximation (also known as the parametric approximation) [11]. We treat the pump field (the field of incoming photons) classically and neglect pump depletion, which is the depletion of the pump power due to losses such as conversion into some alternate optical power that does not contribute to the squeezed light generation. The Hamiltonian of the system reduces to a quadratic form under this approximation.

To model the possible scattering losses of light into the environment, we introduce a phantom channel (as depicted in Figure 1) that has a fictitious coupling to the ring resonator. The Hamiltonian is modified to include the phantom channel by writing $H = H^{\text{L}} + H^{\text{Ph}}$ where H^{Ph} is the contribution from the phantom channel.

5.2 The dynamics of the system

As shown in the previous sections, we can write the time evolution operator in a unique factorised form, $\exp[i\gamma]\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi)$. However, since our Hamiltonian has no single operator terms in $\hat{a}$ and $\hat{a}^\dagger$, the displacement operator does not contribute (with the displacement parameters identically zero), so the factorised form reduces to

$$U(t) = \exp[i\theta(t)]\hat{S}_N(z)\hat{R}_N(\Phi) \tag{55}$$

with the phase rewritten as $\gamma \equiv \theta(t)$.

The process of solving for the evolution operator of the system reduces to determining the squeezing matrix $\mathbf{J}(t)$, the rotation matrix $\boldsymbol{\phi}(t)$ and the phase $\boldsymbol{\theta}(t)$. To this end, it can be shown (manuscript in progress) that in the Heisenberg picture, the system of differential equations we must solve to obtain these elements is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}(t) &= -i\boldsymbol{\Delta}(t)\mathbf{V}(t) - 2i\boldsymbol{\zeta}(t)\mathbf{W}^*(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{W}(t) &= -i\boldsymbol{\Delta}(t)\mathbf{W}(t) - 2i\boldsymbol{\zeta}(t)\mathbf{V}^*(t) \end{aligned} \tag{56}$$

with the initial condition that $\mathbf{V}(t_0)$ is the identity and $\mathbf{W}(t_0)$ is a matrix of zeroes. These matrices satisfy the additional constraints

$$\begin{aligned} 0 &= \mathbf{W}(t)\mathbf{V}^\intercal(t) - \mathbf{V}(t)\mathbf{W}^\intercal(t) \\ 1 &= \mathbf{V}(t)\mathbf{V}^\dagger(t) - \mathbf{W}(t)\mathbf{W}^\dagger(t) \end{aligned} \tag{57}$$

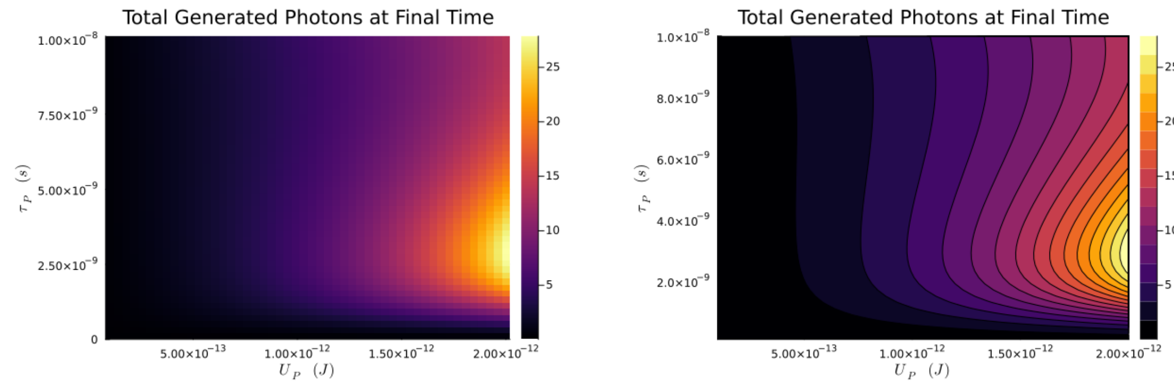
as a direct consequence of the usual annihilation and creation operator commutation relation being preserved by the propagator of the differential equation governing the time evolution of $\hat{a}$ and $\hat{a}^\dagger$ via the given Hamiltonian.

5.3 Simulation results

We implemented a 4th order Runge-Kutta method to solve the system of ordinary differential equations. The results were unpacked using the process described in the manuscript and the elements of the squeezing matrix were used to generate the photon statistics of the outgoing light. The key statistics were the total number of generated photon numbers and the Schmidt number. We describe the utility and nature of these statistics below in context with our results.

5.3.1 Total number of generated photons

The total number of generated photons is the number of photons coming through the output waveguide post the nonlinear optical process occuring in the interaction region of the ring resonator. They comprise the generated squeezed light. This number varies with respect to a range of characteristics for the input pulse. We vary the input pulse power U_P (pJ) and duration τ_P (ns) and keep the other variables fixed (including the coupling efficiency $\eta = \eta_C = \eta_S = \eta_P = 0.5$). The results are formatted as a heatmap with a color gradient scheme indicating the magnitude of the data, and an overlying contour plot is imposed to visualise isoclines of equal magnitude data. See Figure 2.



(a) The raw heatmap showing the number of total generated photons through the ring resonator system with respect to the power of the input pulse U_P and the duration of the input pulse τ_P. The undepleted pump approximation has been made and scattering losses have been modeled by a phantom channel.

(b) An overlaid contour plot over the raw heatmap, showing isoclines for the total number of generated photons with respect to U_P and τ_P. We note the presence of a 'sweet spot' that indicates a maximisation of the number of total generated photons (approximately 27.9) corresponding to around $U_P \approx 2.0$ pJ and $\tau_P \approx 2.93$ ns.

Figure 2: The raw heatmap and overlaid contour plot for the total number of generated photons with respect to input pulse power U_P and input pulse duration τ_P ranging over 0.10 - 2.00 pJ and 0.10 - 10.00 nS respectively.

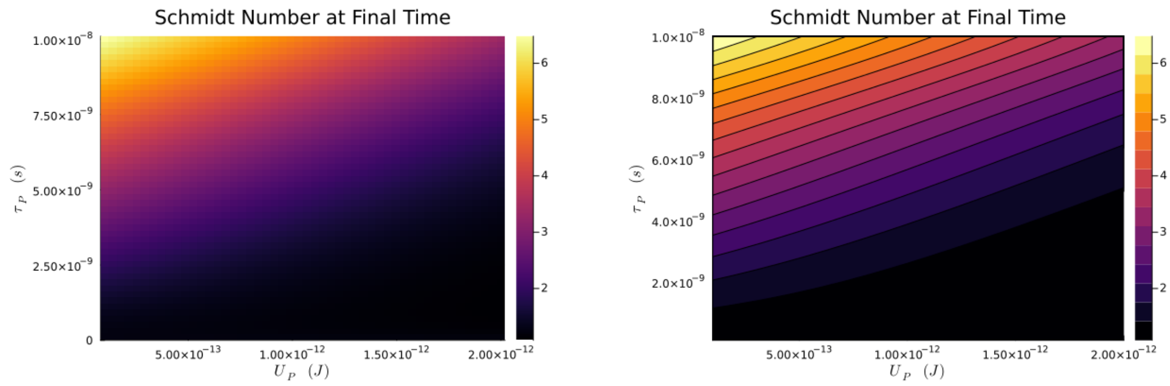
The existence of the sweet spot is expected; for a very short pulse duration, the width of the pulse in frequency space is much larger than the width of the ring resonance. This results in only a small fraction of the input pulse entering the ring. For fixed pulse energy, the power of the pulse decreases with respect to increasing pulse duration. As the duration increases, more of the pulse is sent into the ring but the decreased power results in a reduction in the number of photons generated.

5.3.2 Schmidt number

The Schmidt number K "characterises the effective number of spectral or temporal modes" in the Schmidt decomposition as stated in [12]. It is given by

$$K = \frac{(\sum_\lambda \sinh^2(r_\lambda))^2}{\sum_\lambda \sinh^4(r_\lambda)} \tag{58}$$

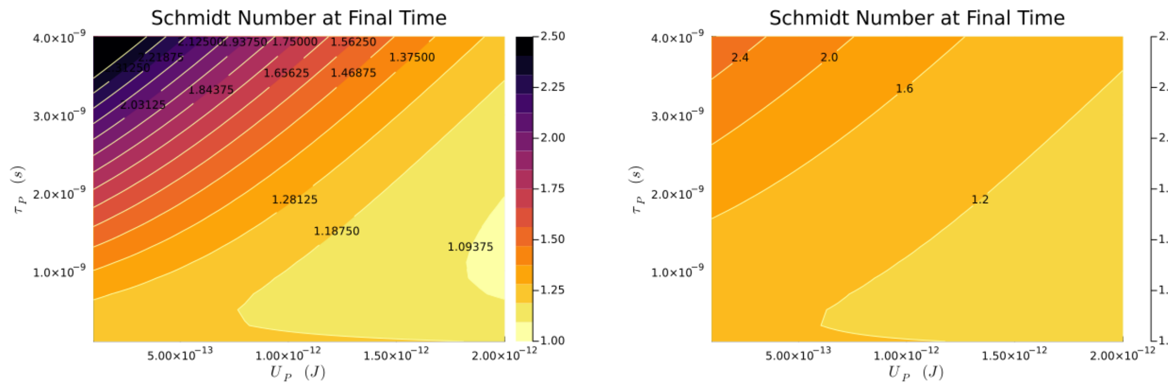
where r_λ are the squeezing parameters associated with the Schmidt modes [13]. It is a measure of the purity of the produced states, i.e. a measure of the quality of entanglement. We refer the reader to [12] for a full description. It is desirable to minimise this number to $K \sim 1.0$ (which would correspond to increased purity) when generating squeezed light as this leads to squeezing which is effectively single mode. Moreover, Schmidt numbers close to 1.0 are desirable in the weak squeezing limit where only a pair of photons are generated since this corresponds to the creation of heralded single photons. See Figures 3 and 4.



(a) The raw heatmap showing the Schmidt number with respect to the power of the input pulse U_P and the duration of the input pulse τ_P. The undepleted pump approximation has been made and scattering losses have been modeled by a phantom channel.

(b) An overlaid contour plot over the raw heatmap, showing isoclines for the Schmidt number. We show the region of interest in this figure (corresponding to K values approaching 1.0) in Figure 4.

Figure 3: The raw heatmap and overlaid contour plot for the Schmidt number with respect to input pulse power U_P and input pulse duration τ_P ranging over 0.10 - 2.00 pJ and 0.10 - 10.00 nS respectively.



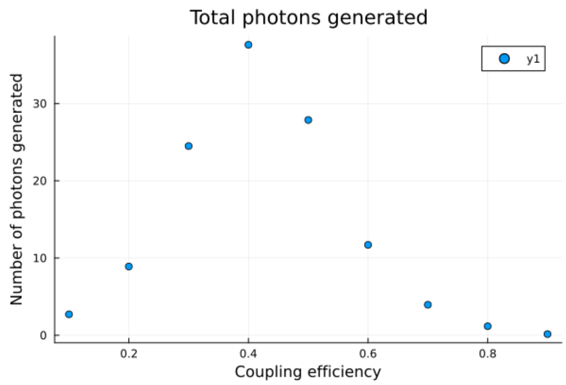
(a) A reversed color-scheme contour plot with a focus on the region of $K \sim 1.0$.

(b) The contour plot in (a) with further magnification.

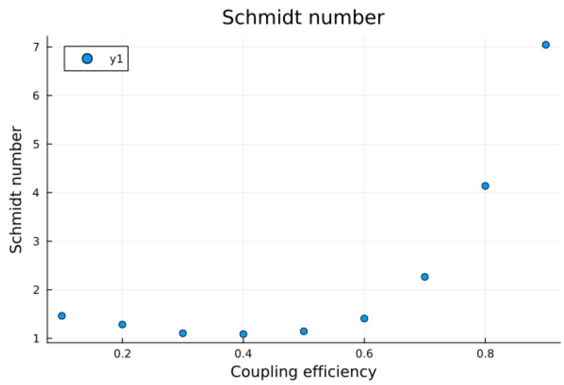
Figure 4: Reverse color-scheme contour plots displaying the region of minimsed Schmidt number in greater detail. The contour plot in (a) is produced via magnification on the region of minimised Schmidt number (isoclines close to $K = 1.0$). The reversed color scheme reveals a similar isocline pattern to the plot in Figure 2(b), with the sweet spot area corresponding to a similar area of minimised Schmidt number, as expected given the relationship between the two statistics. We show the region of interest in greater detail in (b).

5.3.3 Real channel coupling efficiency

The channel waveguide and ring resonator couple via the overlap of the evanescent fields associated with the ring and the channel. The coupling of the channel waveguide to the ring resonator is indicated in Figure 1. The efficiency of this coupling ranges from 0.01 to 0.99 in our simulation (with efficiency under 0.50 resulting in undercoupling and efficiency over 0.50 resulting in overcoupling). See Figure 5. For a complete description of the mechanism of coupling, see [10] and [8].



(a) Plot of the total number of photons generated with respect to the real coupling efficiency $\eta = \eta_C = \eta_S = \eta_P$. We see a peak in the number of total number of photons generated at $\eta = 0.4$.



(b) Plot of the Schmidt number with respect to the real coupling efficiency $\eta = \eta_C = \eta_S = \eta_P$. We see a minimum close to $K = 1$ for $\eta = 0.4$.

Figure 5: Plots showing the variation in the total number of photons generated and the Schmidt number with respect to the coupling efficiency η. The values of U_P and τ_P are fixed to be those corresponding to the sweet spot observed in Figure 2(b). With the peak in the total number of photons generated and the minimum in the Schmidt number both corresponding to $\eta = 0.4$ (as expected given the relation between the total number of photons generated and the Schmidt number), the results agree with [14].

6 Conclusion

We have reconstructed a proof for the unique factorised representation of the time evolution operator for an N-mode quadratic Hamiltonian due to [1], providing the necessary Lie algebraic background and operator calculus methods for normal ordering and disentangling. We applied this result to the process of generating squeezed light via a coupled channel waveguide - ring resonator system with an effective $\chi^{(2)}$ nonlinearity and examined the variation in the total number of photons generated and the Schmidt number for a range of input pulse energies U_P and durations τ_P. We identified the optimal parameter values within these ranges that maximise the total number of generated photons and minimise the Schmidt number. Using these values, we determined the optimal real waveguide-to-ring coupling efficiency η.

The fact that such a unique form for the time evolution operator is possible is very useful due to the ubiquity of quadratic Hamiltonians governing squeezed light generation within the context of integrated photonic structures. Although the original proof in [1] is from 1990, we find that the emergence of new methods incorporating integrated photonic structures (such as ring resonator systems) has resulted in a necessary reexamination of the theoretical underpinnings of modern quantum optics, with direct links to applications and simulations of such systems.

7 Appendix A

7.1 Lie-algebraic background to the disentangling of operators

In section 2, we noted that the authors define operators forming an infinite dimensional Lie algebra and then, via the commutators of these operators with the others, understood that the Lie algebra is homomorphic to su(1,1). The action of the homomorphism on the defined operators led to the rise of a new set of algebraic elements satisfying given commutation relations that ultimately allow for the factorisation of a specific matrix that is very useful in our efforts to disentangle $\hat{S}_N$. In this appendix, we provide a proof for this (which the authors do not provide).

7.1.1 Defining $\hat{A}(u)$, $\hat{A}^\dagger(v)$, $\hat{B}(w)$ and their infinite-dimensional Lie algebra

The authors define the operators $\hat{A}(u)$, $\hat{A}^\dagger(v)$, and $\hat{B}(w)$ as follows:

$$\hat{A}(u) := \frac{1}{2}\hat{a}^\intercal u^\dagger \hat{a} \tag{59}$$

$$\hat{A}^\dagger(v) := \frac{1}{2}\hat{a}^{\dagger\intercal} v \hat{a}^\dagger \tag{60}$$

$$\hat{B}(w) := \frac{1}{2}(\hat{a}^{\dagger\intercal} w \hat{a} + \hat{a}^\intercal w^\intercal \hat{a}^\dagger) \tag{61}$$

where u, v, and w are $N \times N$ matrices. Using the BCH formula, the authors show that

$$\exp\Big[-\hat{B}(w)\Big]\hat{A}(u)\exp\Big[\hat{B}(w)\Big] = \hat{A}(u)\left(\exp[w^\dagger]u\exp[w^{\intercal\dagger}]\right) \tag{62}$$

$$\exp\Big[\hat{B}(w)\Big]\hat{A}^\dagger(v)\exp\Big[-\hat{B}(w)\Big] = \hat{A}^\dagger\left(\exp[w]v\exp[w^\intercal]\right) \tag{63}$$

The authors then state that the operator set

$$\{\hat{A}(u), \hat{A}^\dagger(v), \hat{B}(w) | u, v = (zz^\dagger)^n z, w = (zz^\dagger)^m\}, \quad \text{with } n = 0, 1, \ldots, \text{ and } m = 1, 2, \ldots \tag{64}$$

with $z = z^\intercal$ forms a Lie algebra $\mathcal{L}$ of infinite dimension, with the trivial Lie bracket:

$$[\hat{A}(u), \hat{A}(v)] = [\hat{A}^\dagger(u), \hat{A}^\dagger(v)] = [\hat{B}(w), \hat{B}(w')] = 0. \tag{65}$$

Proof. We see that the vector space formed by (64) consists of a countably infinite number of vectors since the operators take on a countably infinite number of inputs that are indexed by the powers involved $n = 0, 1, \ldots$ and $m = 1, 2, \ldots$, i.e. the powers form an index for the input vectors *in addition to* their usual function as powers acting on functions of the parameter z. Hence, since the vector space is infinite dimensional, so is the Lie algebra. The fact that the vector space forms a Lie algebra is demonstrated by the authors, by showing the Lie bracket of any two vectors in this vector space satisfies the properties of the Lie bracket. □

7.1.2 The homomorphism ψ and matrix representations

This Lie algebra is homomorphic to the Lie algebra su(1,1), via the homeomorphism $\psi\colon \mathcal{L} \to$ su$(1,1)$[13]:

$$\psi[\hat{A}(u)] = L_-, \quad \psi[\hat{A}^\dagger(v)] = L_+, \quad \psi[\hat{B}(w)] = 2L_0 \tag{66}$$

where these operators satisfy the following commutation relations:

$$[L_-, L_+] = 2L_0, \quad [L_0, L_\pm] = \pm L_\pm. \tag{67}$$

[13] See [15] for more information on the ubiquity of su(1,1) in quantum optics.

The representation of $\mathcal{L}$ is the homomorphism ψ. It is faithful, i.e. the kernel (the part of the domain of ψ that maps to the zero vector) is $\{0\}$ itself. Simply put, this means that the transformation is injective. The matrix form of the representation is the following:

$$A(u) \equiv \begin{bmatrix} 0 & 0 \\ -u^\dagger & 0 \end{bmatrix}, \quad A^\dagger(v) \equiv \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}, \quad B(w) \equiv \begin{bmatrix} w & 0 \\ 0 & -w^\intercal \end{bmatrix} \tag{68}$$

where, as we recall, u, v and w are $N \times N$ matrices. Thus, the dimension of the matrices representing the elements of $\mathcal{L}$ is $2N \times 2N$. They must satisfy the same commutation rules that the operator set satisfied.

7.1.3 Matrix factorisation underlying disentangling

Recall that $z = r \exp[i\theta] = \exp[i\theta^\intercal] r^\intercal$ where r was the squeeze parameter (PD/PSD Hermitian matrix, i.e. with nonnegative finite elements) and θ was the squeezing angle (Hermitian matrix with elements $0 \leqslant [\theta]_{ij} \leqslant 2\pi$). We want to factorise the matrix $\exp\big[A^\dagger(z) - A(z)\big]$; as defined by the first equation in (15), $\exp\big[A^\dagger(z) - A(z)\big]$ represents $\hat{S}_N(z)$.

$$\exp\big[A^\dagger(z) - A(z)\big] = \exp\left[\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}\right] \tag{69}$$

by the matrix representations of $A^\dagger$ and A in (68). The matrix exponential is defined as a power series

$$\exp[A] = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{70}$$

Expanding $\exp\left[\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}\right]$ using this definition, we find that the zeroth order term is

$$\frac{\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}^0}{0!} = I \tag{71}$$

the 2×2 identity matrix. The first order term is

$$\frac{\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}^1}{1!} = \begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix} \tag{72}$$

The second order term is

$$\frac{\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}^2}{2!} = \frac{1}{2}\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} zz^\dagger & 0 \\ 0 & z^\dagger z \end{bmatrix} \tag{73}$$

The third order term is

$$\frac{\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}^3}{3!} = \frac{1}{6}\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & zz^\dagger z \\ z^\dagger z z^\dagger & 0 \end{bmatrix} \tag{74}$$

Hence, up to the third order term, the Taylor expansion of the matrix exponential is

$$\exp\left[\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}\right] = I + \begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} zz^\dagger & 0 \\ 0 & z^\dagger z \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 0 & zz^\dagger z \\ z^\dagger z z^\dagger & 0 \end{bmatrix} + h.o.t. \tag{75}$$

Using the definition of z as $z = r\exp[i\theta] = \exp[i\theta^\intercal]r^\intercal$, we can write the expansion in terms of the squeezing parameter and squeezing angle:

$$\exp\left[\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}\right] = I + \begin{bmatrix} 0 & r\exp[i\theta] \\ \exp[-i\theta]r^\dagger & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} r\exp[i\theta]\exp[-i\theta]r^\intercal & 0 \\ 0 & (r^\intercal)^\dagger\exp[-i\theta^\intercal]\exp[i\theta^\intercal]r^\intercal \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 0 & r\exp[i\theta]\exp[-i\theta]r^\dagger r\exp[i\theta] \\ (r^\intercal)^\dagger\exp[-i\theta^\intercal]\exp[i\theta^\intercal]r^\intercal(r^\intercal)^\dagger\exp[-i\theta^\intercal] & 0 \end{bmatrix} + h.o.t. \tag{76}$$

Simplifying by recalling that $r^\dagger = r$ due to r being hermitian[14], we get

$$\exp\left[\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}\right] = I + \begin{bmatrix} 0 & r\exp[i\theta] \\ \exp[-i\theta]r & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} r^2 & 0 \\ 0 & (r^\intercal)^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 0 & r^3\exp[i\theta] \\ rr^\intercal r\exp[-i\theta^\intercal] & 0 \end{bmatrix} + h.o.t \tag{77}$$

In general, the Nth term of the expansion will consist of even functions of r:

$$\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}^N = \begin{bmatrix} r^N & 0 \\ 0 & (r^\intercal)^N \end{bmatrix} \tag{78}$$

as $(-r)^N$ for even N is r^N. Then, $N+1$ will be an odd power and accordingly the $N+1$th term of the expansion will consist of odd functions;

$$\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}^{N+1} = \begin{bmatrix} 0 & r^{N+1}\exp[i\theta] \\ (rr^\intercal)^N r\exp[-i\theta^\intercal] & 0 \end{bmatrix} \tag{79}$$

as $(-r)^{N+1} = -r^{N+1}$. This continues for $N+2$ (even), $N+3$ (odd), and so on. Taking the individual terms of the expansion of the exponential matrix into a single 2×2 matrix, the matrix elements will be:
In the (1,1) position:

$$1 + \frac{1}{2!}r^2 + \frac{1}{4!}r^4 + \cdots + \frac{1}{N!}r^N. \tag{80}$$

In the (1,2) position:

$$r\exp[i\theta] + \frac{1}{3!}r^3\exp[i\theta] + \cdots + \frac{1}{(N-1)!}r^{N-1}\exp[i\theta]. \tag{81}$$

In the (2,1) position:

$$\exp[-i\theta]r^\intercal + \frac{1}{3!}(rr^\intercal)^2 r\exp[-i\theta] + \ldots \frac{1}{(N-1)!}(rr^\intercal)^{N-1}r\exp[-i\theta]. \tag{82}$$

In the (2,2) position:

$$1 + \frac{1}{2!}(r^\intercal)^2 + \cdots + \frac{1}{N!}(r^\intercal)^N \tag{83}$$

assuming we are considering expansion to powers up to N.

[14]The simplified expressions for the elements of these matrices will differ based on whether we use the $z = r\exp[i\theta]$ definition or the $z = \exp[i\theta^\intercal]r^\intercal$ definition. I have used them interchangeably here based on whatever lead to the most simplified expressions, so even if the expressions will look different if you use the alternative definition to what I used, rest assured that they are equivalent.

Now, we see that the power series in the (1,1) position is exactly the Taylor expansion of $\cosh(r)$, the power series in the (1,2) position is exactly the Taylor expansion of $\sinh(r)\exp[i\theta]$, the power series in the (2,1) position is exactly the power series of $\exp[-i\theta]\sinh(r)$ and the power series in the (2,2) position is exactly the Taylor expansion of $\cosh(r^\intercal)$. Hence, we have the following:

$$\exp\left[\begin{bmatrix} 0 & z \\ z^\dagger & 0 \end{bmatrix}\right] = \begin{bmatrix} \cosh(r) & \sinh(r)\exp[i\theta] \\ \exp[-i\theta]\sinh(r) & \cosh(r^\intercal) \end{bmatrix} \tag{84}$$

The authors then state that this is equal to the following decomposition:

$$\begin{bmatrix} I & T \\ 0 & I \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & (S^\intercal)^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ T^\dagger & I \end{bmatrix} \tag{85}$$

where I is the identity matrix, $T := \tanh(r)\exp[i\theta]$, and $S := \operatorname{sech}(r)$. To verify this, we perform the multiplication:

$$\begin{aligned} &\begin{bmatrix} I & T \\ 0 & I \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & (S^\intercal)^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ T^\dagger & I \end{bmatrix} \\ &\quad = \begin{bmatrix} \operatorname{sech}(r) + \tanh(r)\exp[i\theta]\cosh(r)^\intercal\exp[-i\theta]\tanh(r)^\dagger & \tanh(r)\exp[i\theta]\cosh(r)^\intercal \\ \cosh(r)\exp[-i\theta]\tanh(r)^\dagger & \cosh(r)^\intercal \end{bmatrix} \end{aligned} \tag{86}$$

By the definition of z as a symmetric matrix, we see that the decomposition $z = r\exp[i\theta] = \exp[i\theta^\intercal]r^\intercal$ implies

$$f(r)\exp[i\theta] = \begin{cases} \exp[i\theta]f(r^\intercal) & \text{if } f(-r) = f(r) \\ \exp[i\theta^\intercal]f(r^\intercal) & \text{if } f(-r) = -f(r) \end{cases} \tag{87}$$

where f is taken to be expandable as a power series in r. Using this fact,

$$\operatorname{sech}(r) + \tanh(r)\exp[i\theta]\cosh(r)^\intercal\exp[-i\theta]\tanh(r)^\dagger = \operatorname{sech}(r) + \tanh(r)\cosh(r)\tanh(r)$$

which reduces to

$$\frac{1}{\cosh(r)} + \frac{\sinh^2(r)}{\cosh(r)} = \frac{1+\sinh^2(r)}{\cosh(r)} = \frac{1+\cosh^2(r)-1}{\cosh(r)} = \cosh(r) \tag{88}$$

Hence,

$$\operatorname{sech}(r) + \tanh(r)\exp[i\theta]\cosh(r)^\intercal\exp[-i\theta]\tanh(r)^\dagger = \cosh(r). \tag{89}$$

Similarly, the rest of the matrix elements reduce to

$$\tanh(r)\cosh(r)^\intercal = \sinh(r)\exp[i\theta], \tag{90}$$

$$\cosh(r)\exp[-i\theta]\tanh(r)^\dagger = \exp[-i\theta]\sinh(r), \tag{91}$$

$$\cosh(r)^\intercal = \cosh(r^\intercal) \tag{92}$$

using (87). Hence, we have verified the equality between the exponential of the matrix involving z and $z^\dagger$ and the matrix decomposition in terms of I, T, and S. Comparing the matrices in the decomposition with the matrix representation of the operator set (eqn. 68), we see that

$$\begin{bmatrix} I & T \\ 0 & I \end{bmatrix}\begin{bmatrix} S & 0 \\ 0 & (S^\intercal)^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ T^\dagger & I \end{bmatrix} = \exp\big[A^\dagger(T)\big]\exp[B(\ln S)]\exp[-A(T)] \tag{93}$$

and so we have obtained the factorisation of the matrix $\exp\big[A^\dagger(z) - A(z)\big]$ representing the N-mode squeezing operator.

The disentangled form of $\hat{S}_N$ is uniquely determined by the structure of $\mathcal{L}$, i.e. the generators of su(1,1), $L_\pm, L_0$ which we wrote and their commutators with each other ensure that the disentangled form of $\hat{S}_N$ is unique. The disentanging is then performed using the BCH relation as shown in Appendix C.

8 Appendix B

8.1 A note on normal ordering

The concept of normal ordering was first introduced by Gian-Carlo Wick [16] in the context of quantum field theory to avoid the infinities that arose during calculations of field operator expectation values in terms of $\hat{a}^\dagger$ and $\hat{a}$. It has since found extensive use whenever we come across tedious calculations that involve combinations of $\hat{a}^\dagger$ and $\hat{a}$. We reproduce a definition[15] of normal ordering due to C.L. Mehta, [17]:

Remark (Normal ordering notation, $f^{(n)}$ and $:\,:$)**.** *Let $f(\hat{a}, \hat{a}^\dagger)$ be an arbitrary operator function of the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ that obey the usual commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. This commutation relation may be used to put the creation operators occuring within the function to the left of the annihilation operators. When this is the case, the function is said to be in normal ordered form. Denote this normal ordered form by $f^{(n)}(\hat{a}, \hat{a}^\dagger)$. On the other hand, $: f(\hat{a}, \hat{a}^\dagger):$ denotes the form obtained by simply rearranging the annihilation and creation operators without using the commutation relation. By definition,*

$$f^{(n)}(\hat{a}, \hat{a}^\dagger) =: f^{(n)}(\hat{a}, \hat{a}^\dagger): \tag{94}$$

which holds since the colon notation is redundant for a function that is already normal ordered using the commutation relation.

We provide a few elementary examples:

Example 1. *Let $f(\hat{a}, \hat{a}^\dagger) = \hat{a}\hat{a}^\dagger$. Then, $f^{(n)}(\hat{a}, \hat{a}^\dagger) = \hat{a}^\dagger\hat{a} + 1$ and $: f(\hat{a}, \hat{a}^\dagger): \; = \hat{a}^\dagger\hat{a}$.*

Example 2. *In general, $f(\hat{a}, \hat{a}^\dagger) = f^{(n)}(\hat{a}, \hat{a}^\dagger) = \sum_{r,s} f^{(n)}_{rs} \hat{a}^{\dagger r}\hat{a}^s$.*

The second example shows that in the operator sense, the operator functions are equal even though their forms look different. This ordering exists for any operator function that can be expanded as a power series in $\hat{a}$ and $\hat{a}^\dagger$. This is a very useful fact that we have repeatedly used throughout our calculations. We now define the inverse[16] of the normal ordering operator $\mathcal{N}$ (the properties of the normal ordering operator itself are easily read off from the properties of the inverse).

Definition 1 (Inverse of the normal ordering operator, $\mathcal{N}^{-1}$)**.** *The operator $\mathcal{N}^{-1}$ transforms the operator function $f^{(n)}(\hat{a}, \hat{a}^\dagger)$ to an ordinary function $\bar{f}^{(n)}(\alpha, \alpha^*)$ of the complex variable α by simply replacing instances of $\hat{a}$ and $\hat{a}^\dagger$ by α and α^* respectively, i.e. $\mathcal{N}^{-1}[\hat{a}^{\dagger l}\hat{a}^m] = \alpha^{*l}\alpha^m$ where $l, m \in \mathbb{Z}$. It has the following properties:*

1. *Linearity: $\mathcal{N}^{-1}[f_1^{(n)}(\alpha, \alpha^*) + c f_2^{(n)(\alpha,\alpha^*)}] = \bar{f}_1^{(n)}(\hat{a}, \hat{a}^\dagger) + c\bar{f}_2^{(n)}(\hat{a}, \hat{a}^\dagger)$ for $c \in \mathbb{C}$.*
2. *$\mathcal{N}^{-1}[cI] = c$ where I is the identity.*

From these properties, we have

$$\mathcal{N}^{-1}[f^{(n)}(\hat{a}, \hat{a}^\dagger)] = \sum_{r,s} f^{(n)}_{rs} \alpha^{*r}\alpha^s = \bar{f}^{(n)}(\alpha, \alpha^*). \tag{95}$$

By the uniqueness of the normally ordered form, there is a one-to-one correspondence between the normal ordered operator function $f^{(n)}(\hat{a}, \hat{a}^\dagger)$ and the ordinary function $\bar{f}^{(n)}(\alpha, \alpha^*)$. This transformation is explored in greater depth in Appendix E.

[15] The multimode case is easily found by generalising the single mode case.

[16] We introduce the inverse instead of $\mathcal{N}$ itself due to the utility of explicitly showing the properties of a map that takes operator functions to ordinary functions.

8.2 Disentangling and normal ordering the operator product

The general theorem of equation (19) is used to derive the normal ordered form for the operator product $\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi)$:

$$\begin{aligned}\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) = |S|^{\frac{1}{2}} \exp\left[-\frac{1}{2}(\alpha^\dagger\alpha + \alpha^\intercal T^\dagger \alpha)\right] \exp\left[\alpha^\intercal S^\intercal \hat{a}^\dagger + \frac{1}{2}(\hat{a}^\dagger)^\intercal T \hat{a}^\dagger\right] \\ \times \left[\sum_{n=0}^{\infty} \frac{:[(\hat{a}^\dagger)^\intercal (Se^{i\Phi} - I)\hat{a}]^n:}{n!}\right] \exp\left[-(\alpha^\intercal T^\dagger + \alpha^\dagger)e^{i\Phi}\hat{a} - \frac{1}{2}\hat{a}^\intercal e^{i\Phi^\intercal}\hat{a}\right]\end{aligned} \tag{96}$$

We outline the derivation here, beginning with the case of normal ordering just the squeezing operator $\hat{S}_N$. The disentangled form of $\hat{S}_N$ is (18), where we note that there exists a product of three exponential terms. We take the second of these three terms, $\exp\left[(\hat{a}^\dagger)^\intercal(\ln S)\hat{a}\right]$, and find the normal-ordered form of this operator. The authors use the following theorem:

Theorem 1. *For an arbitrary matrix M,*

$$\exp\left[(\hat{a}^\dagger)^\intercal M\hat{a}\right] = \sum_{n=0}^{\infty} \frac{:[(\hat{a}^\dagger)^\intercal(e^M - I)\hat{a}]^n:}{n!} = \sum_{n=0}^{\infty}\sum_{\{n_{ij}\}}\prod_{i,j=1}^{N}\frac{(e^M-I)_{ij}^{n_{ij}}}{n_{ij}!}\prod_{k=1}^{N}\hat{a}_k^{\dagger q_k}\prod_{k=1}^{N}\hat{a}_k^{p_k} \tag{97}$$

where $q_k = \sum_m n_{km}$ and $p_k = \sum_m n_{mk}$. The notation $:\;:$ denotes normal ordering, and $\sum_{\{n_{ij}\}}$ indicates summing over all partitions of $n = \sum_{i,j=1}^{N} n_{ij}$ which are defined to be the individual sums of positive integers to the integer n.[17]

We provide a brief outline of the proof of theorem 1:

Proof. From the BCH formula

$$\exp\Big[\hat{A}\Big]\hat{B}\exp\Big[-\hat{A}\Big] = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$$

we directly see that

$$\begin{aligned}\exp\big[-(\hat{a}^\dagger)^\intercal M\hat{a}\big]\hat{a}\exp\big[(\hat{a}^\dagger)^\intercal M\hat{a}\big] &= \hat{a} + [-(\hat{a}^\dagger)^\intercal M\hat{a},\hat{a}] + \frac{1}{2!}[-(\hat{a}^\dagger)^\intercal M\hat{a},[-(\hat{a}^\dagger)^\intercal M\hat{a},\hat{a}]] + \dots \\ &= \hat{a} - (\hat{a}^\dagger)^\intercal M\hat{a}\hat{a} + \frac{1}{2!}(\hat{a}^\dagger)^\intercal M\hat{a}(\hat{a}^\dagger)^\intercal M\hat{a}\hat{a} - \frac{1}{3!}(\hat{a}^\dagger)^\intercal M\hat{a}(\hat{a}^\dagger)^\intercal M\hat{a}(\hat{a}^\dagger)^\intercal M\hat{a}\hat{a} + \dots \\ &= \hat{a} - (\hat{a}^\dagger)^\intercal M\hat{a} + \frac{1}{2!}(\hat{a}^\dagger)^\intercal M(\hat{a}^\dagger)^\intercal M\hat{a} - \frac{1}{3!}(\hat{a}^\dagger)^\intercal M(\hat{a}^\dagger)^\intercal M(\hat{a}^\dagger)^\intercal M\hat{a} + \dots \\ &= \hat{a} - M\hat{a} + \frac{1}{2!}M^2\hat{a} - \frac{1}{3!}M^3\hat{a} + \dots \\ &= \exp[M]\hat{a}\end{aligned}$$

where the penultimate step is the Taylor expansion of $\exp[M]\hat{a}$. We can rewrite this as

$$[\hat{a},\exp\left[(\hat{a}^\dagger)^\intercal M\hat{a}\right]] = (\exp[M] - I)\exp\big[(\hat{a}^\dagger)^\intercal M\hat{a}\big]\hat{a} \tag{98}$$

Let $F(\hat{a}^\dagger,\hat{a}) = \mathcal{N}\left\{\exp\big[(\hat{a}^\dagger)^\intercal M\hat{a}\big]\right\}$. Recall that N-mode Gaussian squeezed states are defined as

$$|z,\alpha\rangle \equiv \hat{S}_N(z)\,|\alpha\rangle\,, \quad |\alpha\rangle \equiv |\alpha_1\rangle\cdots|\alpha_N\rangle \tag{99}$$

where the latter is an N-mode coherent state (which is a direct product of N single-mode coherent states). It satisfies the eigenvalue equation

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{100}$$

[17]Note that we have the brackets used after exp to be simple square brackets, not the notation for commutators. In every other case, it is generally obvious if the brackets denote a commutator or not.

Then, we can write the inner products

$$\langle\alpha|\exp\big[(\hat{a}^\dagger)^\intercal M\hat{a}\big]|\beta\rangle = \langle\alpha|\beta\rangle F(\alpha^*,\beta) \tag{101}$$

$$\langle\alpha|\,[\hat{a},\exp\big[(\hat{a}^\dagger)^\intercal M\hat{a}\big]]\,|\beta\rangle = \langle\alpha|\beta\rangle \frac{\partial}{\partial\alpha^*}F(\alpha^*,\beta) \tag{102}$$

Together with the commutator relation (98), these inner products yield the initial value problem

$$\frac{\partial}{\partial\alpha^*}F(\alpha^*,\beta) = (\exp[M]-I)F(\alpha^*,\beta)\beta, \quad F(0,\beta)=1 \tag{103}$$

Integrating with respect to α^* to solve the differential equation, we find that

$$\begin{aligned} F(\alpha^*,\beta) &= \exp\big[\alpha^\dagger(\exp[M]-I)\beta\big] && (104)\\ &= \sum_{n=0}^{\infty}\frac{[\alpha^\dagger(\exp[M]-I)\beta]^n}{n!} && (105)\\ &= \sum_{n=0}^{\infty}\sum_{\{n_{ij}\}}\prod_{i,j=1}^{N}\frac{(\exp[M]-I)_{ij}^{n_{ij}}}{n_{ij}!}\prod_{k=1}^{N}(\alpha_k^*)^{q_k}\prod_{k=1}^{N}\beta_k^{p_k} && (106)\end{aligned}$$

and this is exactly what was desired: the normal ordered form of the exponential operator $\exp\big[(\hat{a}^\dagger)^\intercal M\hat{a}\big]$. Applying this to the disentangled form of the N-mode squeeze operator (18), the authors find the normal ordered form (20), shown again immediately below.

$$\begin{aligned}\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) = |S|^{\frac{1}{2}}\exp\left[-\frac{1}{2}(\alpha^\dagger\alpha+\alpha^\intercal T^\dagger\alpha)\right]\exp\left[\alpha^\intercal S^\intercal\hat{a}^\dagger+\frac{1}{2}(\hat{a}^\dagger)^\intercal T\hat{a}^\dagger\right]\\ \times\left[\sum_{n=0}^{\infty}\frac{:[(\hat{a}^\dagger)^\intercal(Se^{i\Phi}-I)\hat{a}]^n:}{n!}\right]\exp\left[-(\alpha^\intercal T^\dagger+\alpha^\dagger)e^{i\Phi}\hat{a}-\frac{1}{2}\hat{a}^\intercal e^{i\Phi^\intercal}\hat{a}\right]\end{aligned}$$

□

By the properties of normal ordering, we immediately know that the normal ordered form of an operator product is not equivalent (in the operator sense) to the operator product of individual normal ordered operators. Hence, this derivation is much more involved, but it still utilises (104). Recall that the N-mode squeeze, displacement, and rotation operators are defined in (15):

$$\begin{aligned}\hat{S}_N(z) &\equiv \exp\left[\frac{(\hat{a}^\dagger)^\intercal z\hat{a}^\dagger}{2}-\frac{\hat{a}^\intercal z^\dagger\hat{a}}{2}\right]\\ \hat{D}_N(\alpha) &\equiv \exp\left[\alpha^\intercal\hat{a}^\dagger-\alpha^\dagger\hat{a}\right]\\ \hat{R}_N(\Phi) &\equiv \exp\left[i(\hat{a}^\dagger)^\intercal\Phi\hat{a}\right]\end{aligned}$$

As we did for $\hat{S}_N$, we must first disentangle each of them using a BCH relation:

Theorem 2. *If A and B are two noncommuting operators that satisfy the conditions*

$$[A,[A,B]] = [B,[A,B]] = 0$$

then

$$\exp[A+B] = \exp[A]\exp[B]\exp\left[-\frac{1}{2}[A,B]\right] = \exp[B]\exp[A]\exp\left[\frac{1}{2}[A,B]\right] \tag{107}$$

The proof is due to Roy Glauber, as cited in [18].

For $\hat{D}_N$,

$$\hat{D}_N(\alpha) \equiv \exp\left[\alpha^\intercal \hat{a}^\dagger - \alpha^\dagger \hat{a}\right] \tag{108}$$

$$\implies A = \alpha^\intercal \hat{a}^\dagger, B = -\alpha^\dagger \hat{a} \tag{109}$$

We check that the conditions of the theorem apply:

$$\begin{aligned}
[A,[A,B]] &= [\alpha^\intercal \hat{a}^\dagger, [\alpha^\intercal \hat{a}^\dagger, -\alpha^\dagger \hat{a}]] && (110)\\
&= [\alpha^\intercal \hat{a}^\dagger, \alpha^\intercal(-\alpha^\dagger)[\hat{a}^\dagger, \hat{a}]] && (111)\\
&= [\alpha^\intercal \hat{a}^\dagger, \alpha^\intercal(\alpha^\dagger)(I)] && (112)\\
&= [\alpha^\intercal \hat{a}^\dagger, \alpha^\intercal(\alpha^\dagger)] && (113)\\
&= 0 && (114)
\end{aligned}$$

since $\alpha^\intercal \alpha^\dagger = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \begin{bmatrix} \alpha_1^* & \cdots & \alpha_N^* \end{bmatrix}$ is simply a scalar. Hence, by the theorem, we can immediately write

$$\begin{aligned}
\hat{D}_N(\alpha) &\equiv \exp\left[\alpha^\intercal \hat{a}^\dagger - \alpha^\dagger \hat{a}\right] && (115)\\
&= \exp\left[-\alpha^\dagger \hat{a}\right] \exp\left[\alpha^\intercal \hat{a}^\dagger\right] \exp\left[\frac{1}{2}[\alpha^\intercal \hat{a}^\dagger, -\alpha^\dagger \hat{a}]\right] && (116)\\
&= \exp\left[-\alpha^\dagger \hat{a}\right] \exp\left[\alpha^\intercal \hat{a}^\dagger\right] \exp\left[\frac{1}{2}\alpha^\intercal \alpha^\dagger\right] && (117)
\end{aligned}$$

This is the disentangled form of the N-mode displacement operator. For the N-mode rotation operator $\hat{R}_N$,

$$\hat{R}_N(\Phi) \equiv \exp\left[i(\hat{a}^\dagger)^\intercal \Phi \hat{a}\right]$$

we note that it is in the form of a lone exponential factor, so it is already disentangled. The product of the three disentangled forms of $\hat{S}_N, \hat{R}_N$, and $\hat{D}_N$ is

$$\begin{aligned}
\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) = |S|^{\frac{1}{2}} &\exp\left[\frac{1}{2}(\hat{a}^\dagger)^\intercal T \hat{a}^\dagger\right] \exp\left[(\hat{a}^\dagger)^\intercal (\ln S)\hat{a}\right] \exp\left[-\frac{1}{2}\hat{a}^\intercal T^\dagger \hat{a}\right]\\
&\times \exp\left[-\alpha^\dagger \hat{a}\right] \exp\left[\alpha^\intercal \hat{a}^\dagger\right] \exp\left[\frac{1}{2}\alpha^\intercal \alpha^\dagger\right] \times \exp\left[i(\hat{a}^\dagger)^\intercal \Phi \hat{a}\right] \qquad (118)
\end{aligned}$$

Comparing to the normal ordered form of the operator product:

$$\begin{aligned}
\hat{S}_N(z)\hat{D}_N(\alpha)\hat{R}_N(\Phi) = |S|^{\frac{1}{2}} &\exp\left[-\frac{1}{2}(\alpha^\dagger \alpha + \alpha^\intercal T^\dagger \alpha)\right] \exp\left[\alpha^\intercal S^\intercal \hat{a}^\dagger + \frac{1}{2}(\hat{a}^\dagger)^\intercal T \hat{a}^\dagger\right]\\
&\times \left[\sum_{n=0}^{\infty} \frac{:[(\hat{a}^\dagger)^\intercal (S e^{i\Phi} - I)\hat{a}]^n:}{n!}\right] \exp\left[-(\alpha^\intercal T^\dagger + \alpha^\dagger) e^{i\Phi}\hat{a} - \frac{1}{2}\hat{a}^\intercal e^{i\Phi^\intercal}\hat{a}\right]
\end{aligned}$$

we see that by repeatedly applying the standard commutation relation for annihilation and creation operators to the disentangled product form and then using the transformation process outlined in Appendix E, the normal ordered form arises. This is an alternate method that the authors do not directly use in their derivation; they opt to use Theorem 1 instead, which leads to the same result more directly. The reason we use the longer (and more tedious) method of going through the process of applying the commutation relation and then the transformation is that it is the same mechanism that the theorem utilises, thus providing us with more insight into the general mathematical technique.

9 Appendix C

9.1 Some essential commutation relations

The following commutator relations are used in combination with the normal ordered form of the time evolution operator to write the transformation of the $\hat{a}$ and $\hat{a}^\dagger$ operators with respect to the time evolution operator.

$$\left[\hat{a}, \hat{U}_N(t)\right] = \hat{a}\hat{U}_N(t) - \hat{U}_N(t)\hat{a} = \frac{\partial}{\partial \hat{a}^\dagger}\hat{U}_N(t), \tag{119}$$

$$\left[\hat{a}^\dagger, \hat{U}_N(t)\right] = \hat{a}^\dagger\hat{U}_N(t) - \hat{U}_N(t)\hat{a}^\dagger = -\frac{\partial}{\partial \hat{a}}\hat{U}_N(t). \tag{120}$$

They follow from a general theorem in [18]. The book does not prove the statement below explicitly, hence we provide a complete proof for it here.

Theorem 3. *Let $f(\hat{a}, \hat{a}^\dagger)$ be a function that is expandable as a power series in $\hat{a}$ and $\hat{a}^\dagger$. Then,*

$$[\hat{a}, f(\hat{a}, \hat{a}^\dagger)] = \frac{\partial f}{\partial \hat{a}^\dagger} \tag{121}$$

and

$$[\hat{a}^\dagger, f(\hat{a}, \hat{a}^\dagger)] = -\frac{\partial f}{\partial \hat{a}} \tag{122}$$

Proof. Since $f = f^{(n)}$, we may expand the derivative of f with respect to $\hat{a}^\dagger$ as

$$\frac{\partial f}{\partial \hat{a}^\dagger} = \frac{\partial f^{(n)}}{\partial \hat{a}^\dagger} = \frac{\partial}{\partial \hat{a}^\dagger}\left(\sum_{j,k} f^{(n)}_{jk}\hat{a}^{\dagger j}\hat{a}^k\right) \tag{123}$$

by expanding $f^{(n)}$ as a power series in $\hat{a}^\dagger$ and $\hat{a}$. Differentiating, we find

$$\frac{\partial f}{\partial \hat{a}^\dagger} = \sum_{j,k} f^{(n)}_{jk} j\hat{a}^{\dagger(j-1)}\hat{a}^k \tag{124}$$

Now, expanding the commutator $[\hat{a}, f(\hat{a}, \hat{a}^\dagger)]$,

$$[\hat{a}, f(\hat{a}, \hat{a}^\dagger)] = \sum_{j,k} f^{(n)}_{jk}[\hat{a}, \hat{a}^{\dagger j}\hat{a}^k] \tag{125}$$

As the annihilation and creation operators are noncommuting, we can expand the commutator in the sum using the identity

$$[A, BC] = [A, B]C + B[A, C] \tag{126}$$

which yields

$$\sum_{j,k} f^{(n)}_{jk}\left([\hat{a}, \hat{a}^{\dagger j}]\hat{a}^k + \hat{a}^{\dagger j}[\hat{a}, \hat{a}^k]\right) \tag{127}$$

The second commutator evaluates to 0. The first commutator, by Theorem 1 in section 3.3 of [18], evaluates to

$$[\hat{a}, \hat{a}^{\dagger j}]\hat{a}^k = j\hat{a}^{\dagger(j-1)}\hat{a}^k \tag{128}$$

and so

$$[\hat{a}, f(\hat{a}, \hat{a}^\dagger)] = \sum_{j,k} f^{(n)}_{jk}[\hat{a}, \hat{a}^{\dagger j}\hat{a}^k] = \sum_{j,k} f^{(n)}_{jk} j\hat{a}^{\dagger(j-1)}\hat{a}^k \tag{129}$$

which is precisely what we obtained in equation (124). □

The proof for the second commutation relation follows similarly. By this general result, the commutation relations (119) and (120) follow immediately.

10 Appendix D

10.1 Time differentiation of the normal ordered time evolution operator

The normal ordered time evolution operator is a product of exponential operators:

$$\hat{U}_N(t) = \exp\left[A(t)\right]\exp\left[B^\intercal(t)\hat{a}^\dagger + (\hat{a}^\dagger)^\intercal C(t)\hat{a}^\dagger\right]\left[\sum_{n=0}^{\infty}\frac{:[(\hat{a}^\dagger)^\intercal D(t)\hat{a}]^n:}{n!}\right]\exp\left[E^\intercal(t)\hat{a} + \hat{a}^\intercal F(t)\hat{a}\right]$$

The time differentiation:

$$i\frac{\partial}{\partial t}\hat{U}_N(t) = i\frac{\partial}{\partial t}\left(\exp\left[A(t)\right]\exp\left[B^\intercal(t)\hat{a}^\dagger + (\hat{a}^\dagger)^\intercal C(t)\hat{a}^\dagger\right]\left[\sum_{n=0}^{\infty}\frac{:[(\hat{a}^\dagger)^\intercal D(t)\hat{a}]^n:}{n!}\right]\exp\left[E^\intercal(t)\hat{a} + \hat{a}^\intercal F(t)\hat{a}\right]\right) \tag{130}$$

In this normal ordered form, the time evolution operator can be ordered according to a transformation such that the operator function that satisfies the evolution equation (26) is in unique correspondence with an equivalent function of commuting variables, thus allowing the operator differential equation (26) to be equivalent to an algebraic differential equation that can be satisfied by ordinary means [19]. Once a solution to the algebraic differential equation is obtained, the inverse of the original transformation may be applied to yield the operator function that solves (26). Note the following theorem from [18]:

Theorem 4. *If m is an integer and $f^{(n)} = f^{(a)} = f$ (where the superscript (n) indicates normal ordered form and (a) indicates antinormal ordered form), then*

$$(\hat{a})^m f(\hat{a},\hat{a}^\dagger) = \mathcal{N}\left[\left(\alpha + \frac{\partial}{\partial\alpha}\right)^m f^{(n)}(\alpha,\alpha^*)\right] = \mathcal{N}\left[\langle\alpha|\,(\hat{a})^m f(\hat{a},\hat{a}^\dagger)\,|\alpha\rangle\right] \tag{131}$$

$$f(\hat{a},\hat{a}^\dagger)(\hat{a}^\dagger)^m = \mathcal{N}\left[\left(\alpha^* + \frac{\partial}{\partial\alpha^*}\right)^m f^{(n)}(\alpha,\alpha^*)\right] = \mathcal{N}\left[\langle\alpha|\,f(\hat{a},\hat{a}^\dagger)(\hat{a}^\dagger)^m\,|\alpha\rangle\right] \tag{132}$$

where $\mathcal{N}$ denotes the normal ordering operator.

There is a **major note of caution** regarding notation here: the variable α used in the statement of this theorem is NOT the same as the coherent state variable α used in writing $\hat{D}_N(\alpha)$. The α as used in the statement of this theorem is simply a complex variable in ordinary function space, i.e. a complex algebraic variable. It is used in the sense that we are using the unique correspondence[18] between an operator function in $\hat{a}$ and $\hat{a}^\dagger$ and an ordinary function in α,α^* to solve the Schrödinger equation, as used in [19]. This caution also applies for the next appendices; it is an abuse of notation but it is also the convention used throughout the works cited, so we adhere to it for the sake of reference.

Using this theorem, we can write the term

$$\hat{H}_N(\hat{a},\hat{a}^\dagger,t)\hat{U}_N(t) = \Xi(\alpha,\alpha^*,t)\times\exp\left[G(\alpha,\alpha^*,t)\right] \tag{133}$$

where the time evolution operator has been written as

$$\hat{U}_N(t) = \exp\left[G(\alpha,\alpha^*,t)\right] \tag{134}$$

[18] This technique is incredibly useful; it resolves the issue of non-commutating variables in operator functions entirely by reformulating our problem in terms of commutating algebraic variables. This new formulation can be attacked using our arsenal of techniques for solving ordinary differential equations, and applying the inverse transformation to our solutions yields the solution of the operator formulation.

where

$$G(\alpha,\alpha^*,t) = A(t) + B^\intercal(t)\alpha^* + (\alpha^*)^\intercal C(t)\alpha^* + (\alpha^*)^\intercal D(t)(\alpha + \frac{\partial}{\partial\alpha^*}) \\ + E^\intercal(t)(\alpha + \frac{\partial}{\partial\alpha^*}) + (\alpha + \frac{\partial}{\partial\alpha^*})^\intercal F(t)(\alpha + \frac{\partial}{\partial\alpha^*}) \quad (135)$$

and the Hamiltonian is denoted by

$$\Xi(\alpha,\alpha^*,t) = (\alpha^*)^\intercal\omega(t)\left(\alpha + \frac{\partial}{\partial\alpha^*}\right) + (\alpha^*)^\intercal f(t)(\alpha^*) + \\ \left(\alpha + \frac{\partial}{\partial\alpha^*}\right)^\intercal f^\dagger(t)\left(\alpha + \frac{\partial}{\partial\alpha^*}\right) + g^\intercal(t)(\alpha^*) + g^\dagger(t)\left(\alpha + \frac{\partial}{\partial\alpha^*}\right) + h(t) \quad (136)$$

The time derivative shown in (54) is then written

$$i\left(\frac{\partial A}{\partial t} + \frac{\partial B^\intercal}{\partial t}\alpha^* + (\alpha^*)^\intercal\frac{\partial C}{\partial t}\alpha^* + (\alpha^*)^\intercal\frac{\partial D}{\partial t}\left(\alpha + \frac{\partial}{\partial\alpha^*}\right) + \frac{\partial E}{\partial t}\left(\alpha + \frac{\partial}{\partial\alpha^*}\right) + \left(\frac{\partial}{\partial\alpha^*}\right)^\intercal\frac{\partial F}{\partial t}\left(\alpha + \frac{\partial}{\partial\alpha^*}\right)\right) \quad (137)$$

Expanding the product in equation (57) and matching terms with the same combination of α, α^*, we can derive the six algebraic differential equations shown in equations (27-32).

11 Appendix E

11.1 The Louisell-Heffner transformation

The underlying mechanism of the calculations done throughout the entire process is the aforementioned unique one-to-one correspondence between an operator function in $\hat{a}, \hat{a}^\dagger$ (i.e. non-commutating operators) and an ordinary function in complex variables α, α^*.[19] This transformation was detailed in [19] and we give a brief overview of their general method.

Given a function $f(\hat{a}, \hat{a}^\dagger)$, we may put it into normal-ordered form which generally looks like

$$f^{(n)}(\hat{a}_i, \hat{a}_i^\dagger) = \sum c_{n_1,n_2,n_3,\ldots,n_k,m_1,m_2,m_3,\ldots,m_i}\hat{a}_1^{\dagger n_1}\hat{a}_2^{\dagger n_2}\ldots\hat{a}_k^{\dagger n_k}\hat{a}_1^{m_1}\hat{a}_2^{m_2}\ldots\hat{a}_i^{m_i} \quad (138)$$

where the coefficients c are functions of time and/or any other parameters involved[20]. Note that as we are using the standard commutation relation $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ to put the function into normal order, the original function and the normal ordered function are equivalent, i.e. $f^{(n)} = f$. Heffner and Louisell define a transformation $T\colon \mathcal{H} \to \mathbb{C}$ where

$$T\colon f(\hat{a}, \hat{a}^\dagger) \mapsto f(\alpha, \alpha^*)$$

where α and α^* are complex variables (and hence commute.) T is applied to an operator function by first putting it into normal order and then substituting $\hat{a}$, $\hat{a}^\dagger$ by α, α^* respectively, i.e.

$$Tf(\hat{a}, \hat{a}^\dagger) = f^{(n)}(\alpha, \alpha^*)$$

Differentiating (138) with respect to time,

$$\frac{\partial f^{(n)}}{\partial t} = \frac{\partial Tf}{\partial t} = \sum\left(\frac{\partial}{\partial t}c_{n_1,n_2,n_3,\ldots,n_k,m_1,m_2,m_3,\ldots,m_i}\right)\hat{a}_1^{\dagger n_1}\hat{a}_2^{\dagger n_2}\ldots\hat{a}_k^{\dagger n_k}\hat{a}_1^{m_1}\hat{a}_2^{m_2}\ldots\hat{a}_i^{m_i} = T\frac{\partial f}{\partial t} \quad (139)$$

[19]The caution in Appendix D is applicable here as well!

[20]The time variable is implicit in the argument of $f^{(n)}$.

By the fact that the normal form of an operator function is unique, T is a unique, one-to-one correspondence between a function of the non-commutating operators and its corresponding ordinary function of commutating complex algebraic variables. In a similar vein, the inverse transformation T^{-1} takes a function of the commuting variables to the same function of the operators in normal form, with all $\hat{a}$ operating to the right.

The gist of the method is that we may write the normal ordered version in commuting complex variables of a given operator function using T, solve the resulting ordinary differential equation (or system of ordinary differential equations) using regular known methods, and subsequently transform the solution(s) into normal ordered operator function(s) using T^{-1}.

To put products of operator functions into normal ordered form, we use Theorem 3 from Appendix C, which can be written as follows: to put $\hat{a}_1 f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i)$ into normal form, we use the following relation:

$$\hat{a}_i f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i) = f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i)\hat{a}_i + \frac{\partial f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i)}{\partial \hat{a}_i^\dagger} \tag{140}$$

which implies

$$[\hat{a}_i f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i)]^{(n)} = \mathcal{N}\left(\left(\hat{a}_i + \frac{\partial}{\partial \hat{a}_i^\dagger}\right) f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i)\right). \tag{141}$$

This generalises to the following: if we have two normal ordered operator functions f and g,

$$[g^{(n)}(\hat{a}_i^\dagger, \hat{a}_i) f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i)]^{(n)} = \mathcal{N}\left(g^{(n)}\left(\hat{a}_i + \frac{\partial}{\partial \hat{a}_i^\dagger}\right) f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i)\right) \tag{142}$$

Thus, by the definition of T and these normal ordering relations, we derive the relations for T applied to products of operator functions:

$$T\left(g(\hat{a}_i^\dagger, \hat{a}_i) f(\hat{a}_i^\dagger, \hat{a}_i)\right) = g^{(n)}\left(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}\right) T f(\hat{a}_i^\dagger, \hat{a}_i) \tag{143}$$

or

$$T\left(g(\hat{a}_i^\dagger, \hat{a}_i) f(\hat{a}_i^\dagger, \hat{a}_i)\right) = f^{(n)}\left(\alpha^*, \alpha + \frac{\partial}{\partial \alpha^*}\right) T g(\hat{a}_i^\dagger, \hat{a}_i) \tag{144}$$

As seen in Appendix D, these are the relations we used to derive the system of six differential equations.

References

[1] X. Ma and W. Rhodes, "Multimode squeeze operators and squeezed states," 1990.

[2] J. Wei and E. Norman, "On global representations of the solutions of linear differential equations as a product of exponentials," 1964.

[3] J. Wei and E. Norman, "Lie algebraic solution of linear differential equations," *Journal of Mathematical Physics*, vol. 4, pp. 575–581, 1963.

[4] W. Magnus, "On the exponential solution of differential equations for a linear operator," *Communications on Pure and Applied Mathematics*, vol. 7, pp. 649–673, 1954.

[5] X. Ma and W. Rhodes, "Squeezing in harmonic oscillators with time-dependent frequencies," *PHYSICAL REVIEW A*, vol. 39, 1989.

[6] P. Lancaster and L. Rodman, "Existence and uniqueness theorems for the algebraic riccati equation," *International Journal of Control*, vol. 32, pp. 285–309, 1980.

[7] D. Martínez-Tibaduiza, C. González-Arciniegas, C. Farina, A. Cavalcanti-Duriez, and A. Z. Khoury, "Quantum-based solution of time-dependent complex riccati equations," *Physical Review A*, vol. 107, 4 2023.

[8] D. G. Rabus, "Integrated ring resonators," 2007.

[9] M. Liscidini, L. G. Helt, and J. E. Sipe, "Asymptotic fields for a hamiltonian treatment of nonlinear electromagnetic phenomena," *Physical Review A - Atomic, Molecular, and Optical Physics*, vol. 85, 1 2012.

[10] N. Quesada, L. G. Helt, M. Menotti, M. Liscidini, and J. E. Sipe, "Beyond photon pairs—nonlinear quantum photonics in the high-gain regime: a tutorial," *Advances in Optics and Photonics*, vol. 14, p. 291, 9 2022.

[11] M. O. Scully and M. S. Zubairy, *Quantum Optics.* Cambridge University Press, 1997.

[12] C. Drago and J. E. Sipe, "Taking apart squeezed light," 10 2023.

[13] M. Houde and N. Quesada, "Waveguided sources of consistent, single-temporal-mode squeezed light: The good, the bad, and the ugly," *AVS Quantum Science*, vol. 5, p. 011404, 02 2023.

[14] M. Banic, L. Zatti, M. Liscidini, and J. E. Sipe, "Two strategies for modeling nonlinear optics in lossy integrated photonic structures," *Phys. Rev. A*, vol. 106, p. 043707, Oct 2022.

[15] M. Novaes, "Some basics of su(1, 1)," 2004.

[16] G. Wick, "The evaluation of the collision matrix," 1950.

[17] C. L. Mehta, "Ordering of the exponential of a quadratic in boson operators. i. single mode case," *Journal of Mathematical Physics*, vol. 18, pp. 404–407, 1976.

[18] W. H. Louisell, *Quantum statistical properties of radiation.* 1990.

[19] H. Heffner and W. H. Louisell, "Transformation having applications in quantum mechanics," *Journal of Mathematical Physics*, vol. 6, pp. 474–478, 1965.