The Louisell-Hefner Transformation

(or the normal-ordering method to solve operator equations)

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These notes are based on the 1965 paper '*Transformation Having Applications in Quantum Mechanics*' by H. Heffner and W. H. Louisell. We explore the foundations and definition of this transformation that allows them to solve equations involving non-commuting operators using ordinary methods known to solve equations of algebraic variables. The foundation of this method is the concept of *operator ordering*, which will be thoroughly covered as an introduction to the method itself. Our primary interest in the transformation lies in its utility to solve the Schrödinger equation for the unitary dynamics defined by any given Hamiltonian. To this end, we explore a few simple examples that show how to employ the method, leading to a basic 'recipe' that makes for a useful reference whenever situations with operator equations demand our attention.

1 Introduction, foundations

By ordering functions of non-commuting operators, we can exploit the unique one-to-one correspondence between operator functions and corresponding functions of commuting algebraic variables to go back and forth between the two. This correspondence allows us to employ conventional methods of solving equations in ordinary variables when dealing with operator equations. One generally applies to transformation to the operator equation, finds a solution to the corresponding ordinary equation, and applies the inverse transformation to yield a solution to the original operator equation.

1.1 Operator ordering

The only difference between operators and commuting variables is that the order in which they are written matters for the former and is of no consequence for the latter. Hence, knowing how to 'order' operators is key to this transformation; this is what lets us associate ordinary functions with their operator-based counterparts.

While the transformation can be applied to any function of non-commuting operators eg. involving quadrature operators $\hat{x}$ and $\hat{p}$, etc. with no practical change in how the resulting calculus works out, we focus on functions of the bosonic creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$. These operators obey the standard bosonic commutation relations:

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij},\ [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = [\hat{a}_i, \hat{a}_j] = 0 \tag{1}$$

It will be useful to establish some key results that set the stage for motivating and describing the transformation. Instead of devoting a full section to these results before displaying the transformation, one can find them laid out in full within the appendices. In this section, I reproduce only the results we will find immediate utility for.

1.1.1 Preliminary definitions and results

We consider functions of $\hat{a}$ and $\hat{a}^\dagger$, $f(\hat{a}_i^\dagger, \hat{a}_i)$. One may define the *normal form* of the function as follows:

Definition 1 (Normal form)**.** *The normal form of a function $f(\hat{a}_i^\dagger, \hat{a}_i)$ is the form of the function where all the creation operators are to the left of the annihilation operators. The normal form of any function $f(\hat{a}_i^\dagger, \hat{a}_i)$ can be written as*

$$f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i) = \sum c_{n_1,\ldots,n_k,m_1,\ldots,m_i} a_1^{\dagger n_1} \ldots a_k^{\dagger n_k} a_1^{m_1} \ldots a_i^{m_i} \tag{2}$$

where the coefficients $c_{n_1,\ldots,n_k,m_1,\ldots,m_i}$ are taken to be complex functions of time or other parameters.

A function $f(\hat{a}_i^\dagger, \hat{a}_i)$ can be put its normal form $f^{(n)}(\hat{a}_i^\dagger, \hat{a}_i)$ by using the bosonic commutation relations (1) to appropriately reorder operators while maintaining the equivalence of the original and normal forms in the operator sense, that is, such that they are both equivalent as operations.

Theorem 1. *Every function of non-commuting operators has a unique normal form.*

Proof. □

2 The transformation

3 Extensions

4 The Schrödinger equation

5 Modern methods

6 Appendices

6.0.1 Useful theorems and lemmas

I source these theorems and the proofs (where stated explicitly) from Ch. 3 of Louisell's 'Quantum Statistical Properties of Radiation'. If the proof for any particular theorem or lemma is not in the chapter, I produce it myself; this is made clear wherever this is necessary. In any case, the proofs are in my own words, laid out as I understand them.