Building a dual pump SFWM interaction

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Contents

1 Setting up the Hamiltonian

1.1 Defining the nonlinear coefficient $K^{nn'}$

We start with the nonlinear SFWM Hamiltonian in the interaction picture:

$$\overline{H}_{\mathrm{NL}}(t) = -\frac{3}{\epsilon_0}\sum_{nn'}\int \mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\, K^{nn'}(k_1,k_2,k_3,k_4)e^{-i\Omega_{nn'}(k_1,k_2,k_3,k_4)t}\hat{a}^\dagger_{nSk_1}\hat{a}^\dagger_{n'Sk_2}b_{Pk_3}b_{Ck_4}+\mathrm{H.c.} \tag{1}$$

where the nonlinear coefficient $K^{nn'}(k_1,k_2,k_3,k_4)$ is given by

$$K^{nn'}(k_1,k_2,k_3,k_4) = \int \mathrm{d}\mathbf{r}\Gamma_3^{ijkl}(\mathbf{r})\left[D^{\mathrm{out},i}_{nSk_1}(\mathbf{r})\right]^*\left[D^{\mathrm{out},j}_{n'Sk_2}(\mathbf{r})\right]^* D^{\mathrm{in},k}_{Pk_3}(\mathbf{r})D^{\mathrm{in},l}_{Ck_4}(\mathbf{r}). \tag{2}$$

We use the formulae for the asymptotic in/out displacement fields in terms of the field enhancement factors, which are given by

$$F_{J-}(k) = \frac{1}{\sqrt{\mathcal{L}}}\left(\frac{\gamma_J^*}{v_J(K_J - k) - i\overline{\Gamma}_J}\right), \qquad J = P, C \tag{3}$$

for the actual input channels and

$$F_{S+}^{(n)}(k) = \frac{1}{\sqrt{\mathcal{L}}}\left(\frac{{\gamma_S^{(n)}}^*}{v_S^{(n)}(K_S^{(n)} - k) + i\overline{\Gamma}_S}\right) \tag{4}$$

for the actual output channels. For simplicity, we assume the coupling constants γ_J, γ_S are real. Then, via equations (A7) - (A9) in Phys. Rev. A 110 033709, the nonlinear coefficient can be rewritten as

$$K^{nn'}(k_1, k_2, k_3, k_4) = \frac{\hbar^2\epsilon_0 v_P v_C}{12\pi^2}\overline{\gamma}_{\mathrm{NL}}\omega_S\mathcal{L}F_{S+}^{(n)*}(k_1)F_{S+}^{(n')*}(k_2)F_{P-}(k_3)F_{C-}(k_4) \tag{5}$$

Plugging in the formulae for the field enhancement factors and simplifying, we find that the nonlinear coefficient is given explicitly by

$$\begin{aligned} K^{nn'}(k_1, k_2, k_3, k_4) = \frac{\hbar^2\epsilon_0 v_P v_C \omega_S \overline{\gamma}_{\mathrm{NL}}}{12\pi^2\mathcal{L}} &\left(\frac{\gamma_S^{(n)}}{v_S^{(n)}(K_S^{(n)} - k_1) - i\overline{\Gamma}_S}\right)\left(\frac{\gamma_S^{(n')}}{v_S^{(n')}(K_S^{(n')} - k_2) - i\overline{\Gamma}_S}\right) \\ &\times \left(\frac{\gamma_P}{v_P(K_P - k_3) - i\overline{\Gamma}_P}\right)\left(\frac{\gamma_C}{v_C(K_C - k_4) - i\overline{\Gamma}_C}\right) \end{aligned} \tag{6}$$

1.2 Defining the detuning parameter $\Omega_{nn'}$

Now for the detuning term. We use the dispersion relation for the channels,

$$\omega_J(k) = \omega_J + v_J(k - K_J) \tag{7}$$

where ω_J is the center frequency of the resonance J, v_J is the group velocity in the channel, and K_J is the wavenumber for the light in either channel with frequency ω_J, i.e. the center of the appropriate resonance. We are given that

$$\Omega_{nn'}(k_1, k_2, k_3, k_4) \equiv \omega_{Ck_4} + \omega_{Pk_3} - \omega_{n'Sk_2} - \omega_{nSk_1}. \tag{8}$$

Using the dispersion relation, we can write the detuning parameter explicitly in terms of the group velocities and wavenumbers:

$$\Omega_{nn'}(k_1, k_2, k_3, k_4) = \omega_C + \omega_P - 2\omega_S + v_C(k_4 - K_C) + v_P(k_3 - K_P) - v_S(k_2 - K_S) - v_S(k_1 - K_S). \tag{9}$$

Now, note that $v_J K_J = \omega_J$, the center frequency. Hence, the expression for $\Omega_{nn'}$ simplifies to

$$\Omega_{nn'}(k_1, k_2, k_3, k_4) = v_C k_4 + v_P k_3 - v_S k_2 - v_S k_1. \tag{10}$$

1.3 Putting $K^{nn'}$ and $\Omega_{nn'}$ together: checking the low-gain solution

We now have the main parts that combine to give us the Λ parameter in the discretized Hamiltonian we are ultimately trying to simulate. First, we find the low-squeezing perturbative solution (the output ket at $t = +\infty$) to see that we are, at least, on the right track to simulating the whole thing *correctly*. The output ket at $t = +\infty$ (considering the initial pulses starts at $t = \infty$ and the interaction in the ring occurs at around $t = 0$) is given by

$$|\psi_\infty\rangle = |\mathrm{vac}\rangle - \left[\frac{i}{\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t' H_{\mathrm{NL}}(t')\right]|\mathrm{vac}\rangle. \tag{11}$$

The only component of the Hamiltonian affecting the time integral is the exponential of the detuning parameter, $e^{-i\Omega_{nn'}t}$. We make use of some the Dirac delta function's many wonderful properties (specifically, the sifting and scaling properties: see Barnett and Radmore's *Methods in Theoretical Quantum Optics*, section A2.8) to do this time integral and ultimately get rid of the integration over k_4.

$$\begin{aligned}\int \mathrm{d}k\, f(k)\delta(\omega(k)) &= \int \mathrm{d}k\, f(k)\delta(vk) \\ &= \int \frac{\mathrm{d}k}{v} f(k)\delta(k) \qquad \text{via the sifting property} \\ &= \int \frac{\mathrm{d}\omega}{v} f\left(\frac{\omega}{v}\delta(\omega)\right) \qquad \text{via } \omega(K) = vK \\ &= \frac{1}{v} f(0)\end{aligned}$$

This means that we can write

$$\int_{-\infty}^{+\infty} \mathrm{d}t\, e^{-i\Omega_{nn'}t} = 2\pi\delta(\Omega) \tag{12}$$

which means that

$$v_C k_4 + v_P k_3 - v_S k_2 - v_S k_1 = 0 \tag{13}$$

and so

$$k_4 = \frac{v_S(k_1 + k_2) - v_P k_3}{v_C} \tag{14}$$

leaving us an integral over k_1, k_2 and k_3.

The SFWM process we are working with involves a dual pump, with two pulses going in with shapes defined as below. These pulse shapes replace the b operators in the original Hamiltonian.

$$\beta_P(k_3) = \sqrt{\frac{N_P v_P \tau_P}{\sqrt{\pi}}} e^{-\frac{1}{2}(k_3 - K_P)^2 v_P^2 \tau_P^2} \tag{15}$$

and

$$\beta_C(k_4) = \sqrt{\frac{N_C v_C \tau_C}{\sqrt{\pi}}} e^{-\frac{1}{2}(k_4 - K_C)^2 v_C^2 \tau_C^2} \tag{16}$$

Using (14), this latter expression becomes

$$\beta_C(k_1, k_2, k_3) = \sqrt{\frac{N_C v_C \tau_C}{\sqrt{\pi}}} \exp\left[-\frac{1}{2v_C^2}\left(v_S(k_1 + k_2) - v_P k_3 - v_C K_C\right)^2 v_C^2 \tau_C^2\right]. \tag{17}$$

Now, consider the parameters $\gamma_J^{(n)}$, which are the coupling constants between a discrete mode of the ring resonator and a continuous waveguide mode of the actual input channel at frequency band $J = S, P, C$. Using the relation

$$|\gamma_J^{(n)}| = \sqrt{2v_J^{(n)}\Gamma_J^{(n)}} = \gamma_J^{(n)} \tag{18}$$

and the fact that $\Gamma_J^{(n)} = \eta_J^{(n)}\overline{\Gamma}_J$, we can state that

$$\gamma_J^{(n)} = \sqrt{\eta_J^{(n)}}\sqrt{2v_J\overline{\Gamma}_J} \tag{19}$$

which we shall handily plug into our integral. This gives us all the components of $\int \mathrm{d}t\, H_{\mathrm{NL}}(t)$.

1.4 The low-gain integral in full

The low-gain integral in full is

$$
\begin{aligned}
-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\, H_{\mathrm{NL}}(t) = -\frac{i\hbar v_P \overline{\gamma}_{\mathrm{NL}}\omega_S v_S \overline{\Gamma}_S \sqrt{N_P N_C v_P^2 v_C^2 \tau_P \tau_C \overline{\Gamma}_P \overline{\Gamma}_C \eta_P \eta_C}}{\pi\sqrt{\pi}\mathcal{L}} \sum_{n,n'} \sqrt{\eta_S^{(n)}}\sqrt{\eta_S^{(n')}} \\
\times \int_{-\infty}^{\infty}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 \, \frac{1}{v_S^{(n)}(K_S^{(n)}-k_1)-i\overline{\Gamma}_S}\frac{1}{v_S^{(n')}(K_S^{(n')}-k_2)-i\overline{\Gamma}_S} \\
\times \frac{1}{v_P(K_P-k_3)-i\overline{\Gamma}_P}\frac{1}{v_S(k_1+k_2)+v_P k_3+i\overline{\Gamma}_C} \\
\times \exp\left[-\frac{1}{2}(k_3-K_P)^2 v_P^2\tau_P^2\right]\exp\left[-\frac{1}{2v_C^2}\left(v_S(k_1+k_2)-v_P k_3-v_C K_C\right)^2 v_C^2\tau_C^2\right] \\
\times \hat{a}^{\dagger}_{nSk_1}\hat{a}_{n'Sk_2} + \mathrm{H.c.} \quad (20)
\end{aligned}
$$

where the sum runs over all possible combinations of the actual channels and phantom channels, labelled *ac* and *ph* respectively. Note that the detuning parameter was absorbed by the time integration that yielded the δ function, in (12). In the full integral, where we move out of the low-gain regime, the detuning parameter will come back into play.

2 Deriving the squeezing matrix J

From our perturbation calculation for the expected squeezed state, we know the following:

$$
|\psi_\infty\rangle = |\mathrm{vac}\rangle - \left[\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\, H_{\mathrm{NL}}(t)\right]|\mathrm{vac}\rangle \quad (21)
$$

Consider the Taylor series expansion of the exponential function:

$$
\exp[X] = \sum_{n=0}^{\infty}\frac{[X]^n}{n!} \approx \mathbb{I} + X \quad (22)
$$

to 2nd order. Then, we can easily identify

$$
|\psi_\infty\rangle = |\mathrm{vac}\rangle - \left[\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\, H_{\mathrm{NL}}(t)\right]|\mathrm{vac}\rangle = \left(\mathbb{I} - \left[\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\, H_{\mathrm{NL}}(t)\right]\right)|\mathrm{vac}\rangle \quad (23)
$$

and so

$$
|\psi_\infty\rangle \approx \exp\left[-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\, H_{\mathrm{NL}}(t)\right]|\mathrm{vac}\rangle\,. \quad (24)
$$

This is equivalent to

$$
\exp\left[\frac{1}{2}\hat{a}^{\dagger\intercal}J\hat{a}^{\dagger} - \mathrm{H.c.}\right]|\mathrm{vac}\rangle \quad (25)
$$

and so

$$
\frac{1}{2}\hat{a}^{\dagger\intercal}J\hat{a}^{\dagger} - \mathrm{H.c.} = -\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\, H_{\mathrm{NL}}(t) = -\frac{i}{\hbar}\hat{a}^{\dagger\intercal}M\hat{a}^{\dagger} + \mathrm{H.c.} \quad (26)
$$

where M is obtained by carrying out the integral of the Hamiltonian over k_3. Thus, we see that

$$
J = -\frac{2i}{\hbar}M \quad (27)
$$

where M is given by

$$M(k_1, k_2) = \int \mathrm{d}k_3 \, K^{nn'}(k_1, k_2, k_3)\beta_P(k_3)\beta_C(k_1, k_2, k_3) + \text{H.c.}$$

where β_P and β_C are defined by (15) and (16) respectively. This is how we obtain the squeezing matrix from the k_3-integral of the nonlinear SFWM Hamiltonian. Keep in mind that this is all done with the integration over time carried out, resulting in the simplification that lets us ignore the k_4-integral – this is possible because we made an approximation with the Taylor series to second order (and because this is really just a check on the Hamiltonian structure to ensure we are actually getting sensible results). The ket this squeezing matrix is associated with is just a perturbative solution and when we actually implement the calculation for the full ket, we cannot ignore k_4 and make such pleasant simplifications to make our lives easier.

3 Escaping the low-gain regime

3.1 The prefactor

It is important for us to get the prefactor right. In this section, I derive it using the contributions from the interaction picture Hamiltonian, nonlinear coefficient, classical pulse shapes, and external factors originating in the change of variables we make to recast the Hamiltonian in terms of position-time instead of momentum-time (as we have maintained so far) and a discretization scheme for the annihilation and creation operators. We discuss the last contribution first.

3.2 Operator discretization

The general theory of discretizing operators for computational purposes comes from QFT, starting with the equations describing the Heisenberg operator, representing the free quantum scalar field $\phi(\mathbf{x}, t)$ and then proceeding with a discretization of the momentum space. The subsequent changes in the volume element (from continuous to discrete) and integral (to a summation over discrete momentum modes) allows us to rewrite the field operator and use the canonical commutation relation between $\hat{a}$ and $\hat{a}^\dagger$ to derive discretized versions of these operators.

Suppose $\hat{a}_\nu$ is the general annihilation operator for a single particle state ν. Let ψ_ν and ψ_ν^* be the ordinary first quantization-derived wavefunction (and its complex conjugate) respectively. The, we can write the second-quantization operators for the quantum field in the real space representation as

$$\Psi(\mathbf{r}) = \sum_\nu \psi_\nu(\mathbf{r})\hat{a}_\nu, \quad \Psi^\dagger(\mathbf{r}) = \sum_\nu \psi_\nu^*(\mathbf{r})\hat{a}_\nu^\dagger \tag{28}$$

These operators are related by the commutation relation $[\Psi(\mathbf{r}_1), \Psi^\dagger(\mathbf{r}_2)] = \delta(\mathbf{r}_1 - \mathbf{r}_2)$. The Heisenberg operator for the scalar field is then written as

$$\phi(\mathbf{x}, t) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{k}}} \left(\hat{a}_\mathbf{k} e^{-ik\cdot x} + \hat{a}_\mathbf{k}^\dagger e^{ik\cdot x}\right) \tag{29}$$

Take a finite volume, say a cube with side length L with volume $V = L^3$. Then, we can discretize momentum space by imposing $k = \frac{2\pi}{L}n$ with $n \in \mathbb{Z}^3$. The volume element d^3k is now discrete, becoming Δk^3. It is related to the volume (and thus the side length) by $\Delta k^3 = \frac{(2\pi)^3}{V} = \left(\frac{2\pi}{L}\right)^3$. Hence, the integral over the infinitesimal volume element now becomes a summation over the discrete volume element,

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3} \to \sum \frac{\Delta k^3}{(2\pi)^3} = \frac{1}{V}\sum_k \tag{30}$$

The scalar field operator can now be rewritten in terms of this new discrete structure:

$$\phi_{\Delta k}(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}_k e^{-ik\cdot x} + \hat{a}_k^\dagger e^{ik\cdot x} \right) \tag{31}$$

where k are now discrete momenta. Recall that the usual commutation relation for the annihilation and creation operators is

$$[\hat{a}(k), \hat{a}^\dagger(k')] = (2\pi)^3 \delta^3(k - k') \tag{32}$$

where δ is the Kronecker delta. The commutator for the respective discrete operators is easily read off of this identity:

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \tag{33}$$

and to ensure consistency with the continuous analog, we introduce a scaling by the square root of the discrete volume element:

$$a_k \to \frac{a_k}{\sqrt{\Delta k^3}} \tag{34}$$

Due to this scaling of the operators, the Kronecker delta for the discrete analog of the usual commutation relation is also appropriately scaled:

$$\delta^3(k - k') \to \frac{\delta_{kk'}}{\Delta k^3} \tag{35}$$

which makes our discrete analog commutator

$$[a_k, a_{k'}^\dagger] \to \frac{1}{\Delta k^3} [a_k, a_{k'}^\dagger] = \frac{\delta_{kk'}}{\Delta k^3}. \tag{36}$$

Thus, our discretized annihilation and creation operators are related to their continuous analogs by

$$a_k = \sqrt{\Delta k^3} \hat{a}(k) \quad \text{or equivalently} \quad \hat{a}(k) = \frac{a_k}{\sqrt{\Delta k^3}} \tag{37}$$

which will now find their place in our Hamiltonian integral. Note that this derivation was done for three-dimensional space, but we will only use the one-dimensional version for our purposes; this is achieved by neglecting the cube over the volume element and letting it remain simply as Δk. This discretization can be visualized by splitting the plot of the resonance into intervals separated by Δk and writing the individual wavenumbers as $k_j = j\Delta k + K_0$ where K_0 is the center of the resonance, and letting j vary from $-n$ to n.

3.3 Change of variables: going from momentum to a dimensionless parameter

We now translate our Hamiltonian from a momentum-time formulation into an integral in terms of a new dimensionless parameter we call x. It is defined as

$$x_J \equiv \frac{v_J(k - K_J)}{\overline{\Gamma}_J} \tag{38}$$

where the index J denotes the resonances S, P, and C, v_J is the group velocity in the channel of the appropriate resonance, and K_J is the center of the resonance as previously noted. The nonlinear coefficient contribution to the integral consists of four terms of the following basic structure:

$$\frac{1}{v_J^{(n)}(K_J^{(n)} - k) - i\overline{\Gamma}_J}. \tag{39}$$

Dividing through by $\overline{\Gamma}_J$, we see that

$$\frac{1}{v_J^{(n)}(K_J^{(n)} - k) - i\overline{\Gamma}_J} = -\frac{1}{\overline{\Gamma}_J(x + i)}. \tag{40}$$

The differential elements $\mathrm{d}k_1$, $\mathrm{d}k_2$, $\mathrm{d}k_3$, and $\mathrm{d}k_4$ thus become $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$, and $\mathrm{d}x_4$. The change of variables results in a contribution to the prefactor:

$$x_J = \frac{v_J(k - K_J)}{\overline{\Gamma}_J} \implies k_J = \frac{\overline{\Gamma}_J x_J}{v_J} + K_J \tag{41}$$

Taking the differential of both sides yields

$$\mathrm{d}k_J = \frac{\overline{\Gamma}_J}{v_J}\mathrm{d}x_J \tag{42}$$

and so there will be a factor of $\frac{\overline{\Gamma}_J}{v_J}$ due to each of the four components making up the nonlinear coefficient; two S resonance terms, a P resonance term, and a C resonance term. The overall contribution to the integral prefactor from these terms is thus

$$\left(\frac{\overline{\Gamma}_S}{v_S}\right)^2 \frac{\overline{\Gamma}_P}{v_P}\frac{\overline{\Gamma}_C}{v_C}. \tag{43}$$

We must also make the change of variables for the detuning parameter. Recall that it was defined in (10). Using (38) and the center frequency equation $v_J K_J = \omega_J$ where $J = S, P, C$, we can write

$$\Omega_{nn'}(x_1, x_2, x_3, x_4) = v_C\left(\frac{\overline{\Gamma}_C x_4}{v_C} + K_C\right) + v_P\left(\frac{\overline{\Gamma}_P x_3}{v_P} + K_P\right) - v_S\left(\frac{\overline{\Gamma}_S x_2}{v_S} + K_S\right) - v_S\left(\frac{\overline{\Gamma}_S x_1}{v_S} + K_S\right) \tag{44}$$

which simplifies to

$$\Omega_{nn'}(x_1, x_2, x_3, x_4) = \overline{\Gamma}_C x_4 + K_C v_C + \overline{\Gamma}_P x_4 + K_P v_P - \overline{\Gamma}_S x_2 - \overline{\Gamma}_S x_1 - 2K_S v_S.$$

Using the center frequency relation, we get

$$\Omega_{nn'}(x_1, x_2, x_3, x_4) = \overline{\Gamma}_C x_4 + \overline{\Gamma}_P x_3 - \overline{\Gamma}_S x_2 - \overline{\Gamma}_S x_1 + \omega_C + \omega_P - 2\omega_S. \tag{45}$$

Note that ω_P, ω_C, and ω_S are the center frequencies of the ring resonances, i.e. they are on resonance with the ring. According to the resonance condition, the circumference of the ring $\mathcal{L} = 2\pi R$ (where R is the ring radius) is an integer multiple of the product of the wavelength of the pulse sent in, λ_C, and the effective refractive index of the pulse mode in the ring $n_{\mathrm{eff}\,C}$:

$$\lambda_C m_C = \mathcal{L} n_{\mathrm{eff}\,C}.$$

The number for the C mode, m_C, is an integer, typically taking on values between 400 to 800. The effective index for the C mode, $n_{\mathrm{eff}\,C}$, is typically 2 for a silicon nitride ring. Identical relations exist for the P and S modes. Consider the quantity

$$\left(\frac{m_C}{n_{\mathrm{eff}\,C}} + \frac{m_P}{n_{\mathrm{eff}\,P}} - \frac{2m_S}{n_{\mathrm{eff}\,S}}\right)\frac{c}{R}.$$

This is generally non-zero since the effective refractive indices are frequency dependent due to chromatic dispersion. Hence, $n_{\mathrm{eff}\,S} \neq n_{\mathrm{eff}\,P} \neq n_{\mathrm{eff}\,C}$. However, if all the frequencies are close, we can safely assume that the effective refractive indices do not change very much with respect to each other, so we can make the approximation $n_{\mathrm{eff}\,S} \approx n_{\mathrm{eff}\,P} \approx n_{\mathrm{eff}\,C}$. The quantity $\left(\frac{m_C}{n_{\mathrm{eff}\,C}} + \frac{m_P}{n_{\mathrm{eff}\,P}} - \frac{2m_S}{n_{\mathrm{eff}\,S}}\right)\frac{c}{R}$ can be rewritten to $\left(\frac{m_C + m_P - 2m_S}{n_{\mathrm{eff}}}\right)\frac{c}{R}$ where $n_{\mathrm{eff}} = n_{\mathrm{eff}\,S} = n_{\mathrm{eff}\,P} = n_{\mathrm{eff}\,C}$. This quantity can be taken to be zero since we can always find three integers m_C, m_P, and m_S satisfying $m_S + m_P = 2m_S$. This approximation is good for the SFWM process, so we can take $\omega_C + \omega_P = 2\omega_S$. Thus, the detuning parameter is, under this approximation,

$$\Omega_{nn'}(x_1, x_2, x_3, x_4) = \overline{\Gamma}_C x_4 + \overline{\Gamma}_P x_3 - \overline{\Gamma}_S(x_2 + x_1) \tag{46}$$

3.4 Building the prefactor

The contributions to the prefactor are the following:

1. From $K^{nn'}$:
$$\frac{\hbar^2\epsilon_0}{12\pi^2\mathcal{L}}v_P v_C\overline{\gamma}_{\mathrm{NL}}\omega_S\gamma_S^{(n)}\gamma_S^{(n')}\gamma_P\gamma_C \tag{47}$$
2. From the interaction picture Hamiltonian prefactor:
$$-\frac{3}{\epsilon_0} \tag{48}$$
3. From the pulse shapes:
$$\sqrt{\frac{N_P v_P\tau_P}{\sqrt{\pi}}}\sqrt{\frac{N_C v_C\tau_C}{\sqrt{\pi}}} \tag{49}$$
4. From the change of variables:
$$\left(\frac{\overline{\Gamma}_S}{v_S}\right)^2\frac{\overline{\Gamma}_P}{v_P}\frac{\overline{\Gamma}_C}{v_C}$$
5. From the operator discretization:
$$\frac{1}{\sqrt{\Delta k_1\Delta k_2}}$$

Putting it all together (and rewriting the $\gamma_J^{(n)}$ using (18)), the final prefactor for our Hamiltonian is
$$-\frac{\hbar^2\overline{\gamma}_{\mathrm{NL}}\omega_S v_P v_C}{\pi^{\frac{5}{2}}\mathcal{L}v_S}\frac{1}{\sqrt{\Delta k_1}}\frac{1}{\sqrt{\Delta k_2}}\overline{\Gamma}_S^3\overline{\Gamma}_P^{\frac{3}{2}}\overline{\Gamma}_C^{\frac{3}{2}}\sqrt{N_P N_C\tau_P\tau_C\eta_P\eta_C} \tag{50}$$

4 The full Hamiltonian

Putting (15) and (16) into (1), changing variables so that the integral is in terms of the dimensionless parameter x (via (38)), and putting the prefactor in place, we have the interaction Hamiltonian:

$$\begin{aligned}
&-\frac{\hbar^2\overline{\gamma}_{\mathrm{NL}}\omega_S v_P v_C\overline{\Gamma}_S^3\overline{\Gamma}_P^{\frac{3}{2}}\overline{\Gamma}_C^{\frac{3}{2}}\sqrt{N_P N_C\tau_P\tau_C\eta_P\eta_C}}{\pi^{\frac{5}{2}}\mathcal{L}v_S\sqrt{\Delta k_1\Delta k_2}}\times\\
&\sum_{n,n'}\sqrt{\eta_S^{(n)}}\sqrt{\eta_S^{(n')}}\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\,\frac{1}{\overline{\Gamma}_S\left(x_1^{(n)}+i\right)}\frac{1}{\overline{\Gamma}_S\left(x_2^{(n')}+i\right)}\frac{1}{\overline{\Gamma}_P\left(x_3+i\right)}\frac{1}{\overline{\Gamma}_C\left(x_4+i\right)}\times\\
&\exp\left[-i\left(\overline{\Gamma}_C x_4+\overline{\Gamma}_P x_3-\overline{\Gamma}_S(x_2+x_1)\right)t\right]\\
&\exp\left[-\frac{1}{2}\overline{\Gamma}_P^2 x_3^2\tau_P^2\right]\exp\left[-\frac{1}{2}\overline{\Gamma}_C^2 x_4^2\tau_C^2\right]a^\dagger_{nSx_1}a^\dagger_{n'Sx_2}+\mathrm{H.c.}
\end{aligned} \tag{51}$$

This simplifies further to:

$$\begin{aligned}
&-\frac{\hbar^2\overline{\gamma}_{\mathrm{NL}}\omega_S v_P v_C\overline{\Gamma}_S\sqrt{\overline{\Gamma}_P N_P\tau_P\eta_P}\sqrt{\overline{\Gamma}_C N_C\tau_C\eta_C}}{\pi^{\frac{5}{2}}\mathcal{L}v_S\sqrt{\Delta k_1\Delta k_2}}\sum_{n,n'}\sqrt{\eta_S^{(n)}}\sqrt{\eta_S^{(n')}}\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\\
&\frac{\exp\left[-\frac{1}{2}\overline{\Gamma}_P^2 x_3^2\tau_P^2\right]\exp\left[-\frac{1}{2}\overline{\Gamma}_C^2 x_4^2\tau_C^2\right]\exp\left[-i\left(\overline{\Gamma}_C x_4+\overline{\Gamma}_P x_3-\overline{\Gamma}_S(x_2+x_1)\right)t\right]}{\left(x_1^{(n)}+i\right)\left(x_2^{(n')}+i\right)(x_3+i)(x_4+i)}\\
&\times a^\dagger_{nSx_1}a^\dagger_{n'Sx_2}+\mathrm{H.c.}
\end{aligned} \tag{52}$$

This is the full simplified Hamiltonian in terms of the dimensionless variable x. We make one last change of variables by introducing the dimensionless 'time' parameter $\bar{t} = t\overline{\Gamma}_S$. Hence, the sole remaining t in the detuning parameter exponential term becomes $\bar{t}/\overline{\Gamma}$, yielding

$$-\frac{\hbar^2\overline{\gamma}_{\mathrm{NL}}\omega_S v_P v_C \overline{\Gamma}_S\sqrt{\overline{\Gamma}_P N_P \tau_P \eta_P}\sqrt{\overline{\Gamma}_C N_C \tau_C \eta_C}}{\pi^{\frac{5}{2}}\mathcal{L}v_S\sqrt{\Delta k_1 \Delta k_2}}\sum_{n,n'}\sqrt{\eta_S^{(n)}}\sqrt{\eta_S^{(n')}}\int \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$$
$$\frac{\exp\left[-\frac{1}{2}\overline{\Gamma}_P^2 x_3^2\tau_P^2\right]\exp\left[-\frac{1}{2}\overline{\Gamma}_C^2 x_4^2\tau_C^2\right]\exp\left[-\frac{i}{\overline{\Gamma}_S}\left(\overline{\Gamma}_C x_4 + \overline{\Gamma}_P x_3 - \overline{\Gamma}_S(x_2+x_1)\right)\bar{t}\right]}{\left(x_1^{(n)}+i\right)\left(x_2^{(n')}+i\right)(x_3+i)(x_4+i)}$$
$$\times\, a^\dagger_{nSx_1}a^\dagger_{n'Sx_2} + \mathrm{H.c.} \quad (53)$$

4.1 Discretizing the integral

We now discretize the integral. For numerical integration purposes, we will be transforming the integral into a sum over discrete values of x_1, x_2, x_3 and x_4, which can be written out as

$$H_{\mathrm{NL}} = -\frac{\hbar^2\overline{\gamma}_{\mathrm{NL}}\omega_S v_P v_C \overline{\Gamma}_S\sqrt{\overline{\Gamma}_P N_P \tau_P \eta_P}\sqrt{\overline{\Gamma}_C N_C \tau_C \eta_C}}{\pi^{\frac{5}{2}}\mathcal{L}v_S\sqrt{\Delta k_1 \Delta k_2}}\Delta x_1\Delta x_2\Delta x_3\Delta x_4\sum_{n,n'}\sqrt{\eta_S^{(n)}}\sqrt{\eta_S^{(n')}}$$
$$\sum_{i=1}^{n_S}\sum_{j=1}^{n_S}\sum_{k=1}^{n_C}\sum_{l=1}^{n_P}$$
$$\times\,\frac{\exp\left[-\frac{1}{2}\overline{\Gamma}_P^2 x_{3,k}^2\tau_P^2\right]\exp\left[-\frac{1}{2}\overline{\Gamma}_C^2 x_{4,l}^2\tau_C^2\right]\exp\left[-\frac{i}{\overline{\Gamma}_S}\left(\overline{\Gamma}_C x_{4,l} + \overline{\Gamma}_P x_{3,k} - \overline{\Gamma}_S(x_{2,j}+x_{1,i})\right)\bar{t}\right]}{\left(x_{1,i}^{(n)}+i\right)\left(x_{2,j}^{(n')}+i\right)(x_{3,k}+i)(x_{4,l}+i)}$$
$$\times\, a^\dagger_{nSx_{1,i}}a^\dagger_{n'Sx_{2,j}} + \mathrm{H.c.} \quad (54)$$

where the grouped indices $i \equiv (x_1, s)$, $j \equiv (x_2, t)$, $k \equiv (x_3, u)$, and $l \equiv (x_4, v)$ have the dimensionless x parameter indices as the first component and a discrete labelling as the second component. We assume the discrete labelling indices i, j, k, l take on values from 1 to N_1, N_2, N_3, N_4 respectively. **Note: the labelling scheme here for indices has nothing to do with the indices we will see in the next section – those come directly from [HSS] whereas the indexing scheme here is entirely self-contained. Do not confuse the two.**

Consider the relationship between the infinitesimal dimensionless x-variable and the original infinitesimal k-variable, given by (42). We can derive the discrete version of the relation to simplify the prefactor further:

$$\mathrm{d}k_J = \frac{\overline{\Gamma}_J}{v_J}\mathrm{d}x_J \implies \Delta k_J = \frac{\overline{\Gamma}_J}{v_J}\Delta x_J \quad (55)$$

which means that

$$\frac{\Delta x_1\Delta x_2\Delta x_3\Delta x_4}{\sqrt{\Delta k_1\Delta k_2}} = \frac{\Delta x_1\Delta x_2\Delta x_3\Delta x_4}{\sqrt{\frac{\overline{\Gamma}_S}{v_S}\Delta x_1\frac{\overline{\Gamma}_S}{v_S}\Delta x_2}} = \frac{v_S}{\overline{\Gamma}_S}\frac{\Delta x_1\Delta x_2\Delta x_3\Delta x_4}{\sqrt{\Delta x_1\Delta x_2}} \quad (56)$$

We will generally require the spacing to be the same for each x-variable with respect to each other. Setting $\Delta x = \Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4$, we obtain

$$\frac{v_S}{\overline{\Gamma}_S}\frac{\Delta x_1\Delta x_2\Delta x_3\Delta x_4}{\sqrt{\Delta x_1\Delta x_2}} = \frac{v_S}{\overline{\Gamma}_S}\frac{(\Delta x)^4}{\sqrt{(\Delta x)^2}} = \frac{v_S}{\overline{\Gamma}_S}(\Delta x)^3 \quad (57)$$

Hence, our prefactor is now

$$-\frac{\hbar^2\overline{\gamma}_{\mathrm{NL}}\omega_S v_P v_C(\Delta x)^3\sqrt{\overline{\Gamma}_P N_P\tau_P\eta_P}\sqrt{\overline{\Gamma}_C N_C\tau_C\eta_C}}{\pi^{\frac{5}{2}}\mathcal{L}}. \tag{58}$$

The full discrete Hamiltonian is thus

$$\begin{aligned}H_{\mathrm{NL}} = &-\frac{\hbar^2\overline{\gamma}_{\mathrm{NL}}\omega_S v_P v_C(\Delta x)^3\sqrt{\overline{\Gamma}_P N_P\tau_P\eta_P}\sqrt{\overline{\Gamma}_C N_C\tau_C\eta_C}}{\pi^{\frac{5}{2}}\mathcal{L}}\sum_{n,n'}\sqrt{\eta_S^{(n)}}\sqrt{\eta_S^{(n')}}\sum_{i=1}^{n_S}\sum_{j=1}^{n_S}\sum_{k=1}^{n_C}\sum_{l=1}^{n_P}\\ &\times\frac{\exp\left[-\frac{1}{2}\overline{\Gamma}_P^2 x_{3,k}^2\tau_P^2\right]\exp\left[-\frac{1}{2}\overline{\Gamma}_C^2 x_{4,l}^2\tau_C^2\right]\exp\left[-\frac{i}{\overline{\Gamma}_S}\left(\overline{\Gamma}_C x_{4,l}+\overline{\Gamma}_P x_{3,k}-\overline{\Gamma}_S(x_{2,j}+x_{1,i})\right)\bar{t}\right]}{\left(x_{1,i}^{(n)}+i\right)\left(x_{2,j}^{(n')}+i\right)(x_{3,k}+i)(x_{4,l}+i)}\\ &\times a^\dagger_{nSx_{1,i}}a^\dagger_{n'Sx_{2,j}}+\mathrm{H.c.}\end{aligned} \tag{59}$$

5 Building the matrix $[\boldsymbol{\zeta}(t)]$

From [HSS], we have that the matrix driving the production of photon pairs in the created squeezed state is

$$[\boldsymbol{\zeta}(t)]_{\mu\nu} = \Lambda_{\mu\nu l}(t)[\boldsymbol{\beta}(t)]_l$$

where $\Lambda_{\mu\nu l}(t)$ is the discretized nonlinear coefficient, consisting of the product of the nonlinear coefficient $K^{nn'}$ and the exponential of the detuning parameter, $\exp[-i\Omega_{nn'}t]$. The discrete indices μ and ν are defined as the groupings of indices (i,n) and (j,n') respectively. μ indicates a photon in the output channel n with wavenumber k_i. The zeta matrix in terms of the discretized nonlinear coefficient and pump pulses is given by

$$\zeta(x_1,x_2,t) = \int \mathrm{d}x_3\mathrm{d}x_4\,\Lambda(x_1,x_2,x_3,x_4,t)\beta_P(x_3)\beta_C(x_4) \tag{60}$$

We can read $\Lambda(x_1,x_2,x_3,x_4,t)$ off the Hamiltonian (53):

$$\begin{aligned}\Lambda^{nn'}(x_1,x_2,x_3,x_4,t) = &-\frac{\hbar\overline{\gamma}_{\mathrm{NL}}\omega_S v_P v_C(\Delta x)^3\sqrt{\overline{\Gamma}_P N_P\tau_P\eta_P}\sqrt{\overline{\Gamma}_C N_C\tau_C\eta_C}}{\pi^{\frac{5}{2}}\mathcal{L}}\sqrt{\eta_S^{(n)}}\sqrt{\eta_S^{(n')}}\\ &\times\frac{\exp\left[-\frac{i}{\overline{\Gamma}_S}\left(\overline{\Gamma}_C x_4+\overline{\Gamma}_P x_3-\overline{\Gamma}_S(x_2+x_1)\right)\bar{t}\right]}{\left(x_1^{(n)}+i\right)\left(x_2^{(n')}+i\right)(x_3+i)(x_4+i)}\end{aligned} \tag{61}$$

Then, $[\boldsymbol{\zeta}(t)]$ is simply the integral of the product of (61) and $\beta_P(x_3)\beta_C(x_4)$ with respect to the indices x_3 and x_4. The reason we have $\hbar$ in the prefactor in (61) instead of $\hbar^2$ as seen in (53) is that our discrete Hamiltonian, similar to the three-indexed one in [HSS], is given by

$$H_{\mathrm{NL}}(t) = \hbar\sum_{n,n'}\sum_{i,j,k,l}\Lambda^{nn'}_{ijkl}(t)a^\dagger_{nSx_{1,i}}a^\dagger_{n'Sx_{2,j}}\beta_P(x_{3,k})\beta_C(x_{4,l})+\mathrm{H.c.}. \tag{62}$$

The zeta matrix is given in terms of x_1, x_2, and t, shown explicitly below:

$$\begin{aligned}\zeta^{nn'}(x_1,x_2,t) = &-\frac{\hbar\overline{\gamma}_{\mathrm{NL}}\omega_S v_P v_C(\Delta x)^3\sqrt{\overline{\Gamma}_P N_P\tau_P\eta_P}\sqrt{\overline{\Gamma}_C N_C\tau_C\eta_C}}{\pi^{\frac{5}{2}}\mathcal{L}}\sqrt{\eta_S^{(n)}}\sqrt{\eta_S^{(n')}}\\ &\times\int \mathrm{d}x_3\mathrm{d}x_4\,\frac{\exp\left[-\frac{i}{\overline{\Gamma}_S}\left(\overline{\Gamma}_C x_4+\overline{\Gamma}_P x_3-\overline{\Gamma}_S(x_2+x_1)\right)\bar{t}\right]}{\left(x_1^{(n)}+i\right)\left(x_2^{(n')}+i\right)(x_3+i)(x_4+i)}\exp\left[-\frac{1}{2}\overline{\Gamma}_P^2 x_3^2\tau_P^2\right]\exp\left[-\frac{1}{2}\overline{\Gamma}_C^2 x_4^2\tau_C^2\right]\end{aligned} \tag{63}$$

The integral is calculated numerically via the discretization procedure we did with the Hamiltonian. We have shown the integral version of the zeta matrix here as it is the 'full' relation; one can easily read off the discretization from this. For completeness, we give the discretized zeta matrix explicitly also:

$$\zeta_{ijkl}^{nn'}(x_1,x_2,t) = -\frac{\hbar^2\overline{\gamma}_{\rm NL}\omega_S v_P v_C(\Delta x)^3\sqrt{\overline{\Gamma}_P N_P\tau_P\eta_P}\sqrt{\overline{\Gamma}_C N_C\tau_C\eta_C}}{\pi^{\frac{5}{2}}\mathcal{L}}\sum_{n,n'}\sqrt{\eta_S^{(n)}}\sqrt{\eta_S^{(n')}}$$
$$\times\sum_{i,j,k,l=1}^{n_J}\frac{\exp\left[-\frac{1}{2}\overline{\Gamma}_P^2x_{3,k}^2\tau_P^2\right]\exp\left[-\frac{1}{2}\overline{\Gamma}_C^2x_{4,l}^2\tau_C^2\right]\exp\left[-\frac{i}{\overline{\Gamma}_S}\left(\overline{\Gamma}_Cx_{4,l}+\overline{\Gamma}_Px_{3,k}-\overline{\Gamma}_S(x_{2,j}+x_{1,i})\right)\bar{t}\right]}{\left(x_{1,i}^{(n)}+i\right)\left(x_{2,j}^{(n')}+i\right)(x_{3,k}+i)(x_{4,l}+i)} \tag{64}$$

where we contracted the four summations into one, with n_J equal to n_S over indices i and j, and equal to n_P and n_C over indices k and l respectively.

The continuous zeta matrix and the discrete zeta matrix are related to each other by the continuous lambda coefficient and the discrete lambda coefficient. One can relate the two coefficients by writing

$$\Lambda_{ijkl}^{nn'}(x_{1,i},x_{2,j},x_{3,k},x_{4,l},\bar{t}) = \frac{v_S}{\overline{\Gamma}_S}(\Delta x)^3\Lambda^{nn'}(x_1,x_2,x_3,x_4,\bar{t}) \tag{65}$$

using the discrete version of the relation for the infinitesimal k and x elements.

5.1 Coupled operator equations for the output field

The Hamiltonian we use for numerical calculations can be written in the form

$$H_{\rm NL}(t) = \hbar\sum_{\mu,\nu}\sum_l\Lambda_{\mu\nu l}(t)a_\mu^\dagger a_\nu^\dagger\beta_P\beta_C + \text{H.c.} \tag{66}$$

which is the same Hamiltonian, in principle, as (62), just slightly simplified index-wise. The unitary time evolution operator for the system, $U(t,t')$, is a solution of the Schrödinger equation:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U(t,t') = HU(t,t') \tag{67}$$

where $U(t',t') = I$ and H is the total Hamiltonian, given by

$$H = H_{\rm L} + H_{\rm NL} \tag{68}$$

We envision the unitary evolution as follows: at some very early initial time t_0, the state's evolution is solely governed by $H = H_{\rm L}$; that is, the contribution of $H_{\rm NL}$ is sufficiently negligible to the overall evolution at t_0 since the state is, physically speaking, very far away from the ring, where $H_{\rm NL}$ is the dominant contribution to H (to the extent that $H_{\rm NL}$ is deemed negligible in the interaction region of the ring). After going into the ring and coming out, the subsequent state is again taken to be solely under the influence of $H_{\rm L}$. For full details on the formalism used to describe the evolution of the input state and (crucially) the initial conditions and effective unitary operators, see [HSS]. We reproduce the essential elements (the ones we will use for numerical implementations) of the story here. Our aim is to derive a set of coupled equations that tell us the story of what comes out of the ring. **This section is completely derived from [HSS].**

5.2 Evolution and Gaussian Parameters

Figure 1. Schematic diagram of the ket evolution process.

The unitary time evolution operator for our system is a solution to the Schrödinger equation,

$$i\hbar\frac{\mathrm{d}U(t,t')}{\mathrm{d}t} = HU(t,t') \tag{69}$$

where the Hamiltonian is

$$H = H_{\mathrm{L}} + H_{\mathrm{NL}} \tag{70}$$

and the initial condition is $U(t',t') = I$ for all times t'. The Hamiltonian is composed of two contributions; H_{L} encapsulates the linear contribution, while H_{NL} encapsulates the nonlinear effects of the ring resonator.

We make the assumption that H_L completely dominates the ket evolution from some initial time t_0 to some time $t_1 < t_{\mathrm{on}}$, where the ket is sufficiently far from the ring, and that H_{NL} is completely dominant on the ket evolution from some time t_{on} to some time $t_{\mathrm{off}} < t_2$. After this time, for a period t_2 to t_3, we have that H_{L} dominates the evolution once more, with H_{NL} making a negligible contribution to the ket evolution as the field excitation has propagated sufficiently far away from the ring.

Consider the evolution of a ket from $t_0 < t_{\mathrm{on}}$ to $t_1 > t_{\mathrm{off}}$. By the composition property of the unitary time evolution operator, we can write the time evolution operator from t_0 to t_1 as the product of the individual time evolution operators from t_0 to t_{on}, t_{on} to t_{off}, and t_{off} to t_1:

$$U(t_1,t_0) = U(t_{\mathrm{on}},t_0)U(t_{\mathrm{off}},t_{\mathrm{on}})U(t_1,t_{\mathrm{off}}). \tag{71}$$

The first and last operators on the right hand side of (71) only involve the linear Hamiltonian, so we may immediately write

$$U(t_1,t_0) = e^{-\frac{i}{\hbar}H_{\mathrm{L}}(t_{\mathrm{on}}-t_0)}U(t_{\mathrm{off}},t_{\mathrm{on}})e^{-\frac{i}{\hbar}H_{\mathrm{L}}(t_1-t_{\mathrm{off}})}. \tag{72}$$

Suppose that at time t_0, an initial ket $|\psi(t_0)\rangle$ is specified. We define the *input* ket $|\psi_{\mathrm{in}}\rangle$:

$$|\psi_{\mathrm{in}}\rangle \equiv e^{-\frac{i}{\hbar}H_{\mathrm{L}}(0-t_0)}\,|\psi(t_0)\rangle \tag{73}$$

which says that the initial ket, starting at $t = t_0$, evolves to $|\psi_{\mathrm{in}}\rangle$ at $t = 0$ assuming $H_{\mathrm{NL}} = 0$. Now, specifying a ket $|\psi(t_1)\rangle$ at a time $t_1 > t_{\mathrm{off}}$, we can similarly define the *output* ket $|\psi_{\mathrm{out}}\rangle$:

$$|\psi_{\mathrm{out}}\rangle \equiv e^{\frac{i}{\hbar}H_{\mathrm{L}}(t_1-0)}\,|\psi(t_1)\rangle\,. \tag{74}$$

The ket at t_1 is given by applying the total time evolution operator $U(t_1, t_0)$ to the ket at t_0:

$$|\psi(t_1)\rangle = U(t_1, t_0)\,|\psi(t_0)\rangle\,. \tag{75}$$

which we can rewrite in terms of the input ket and the output ket as follows: defining the operator

$$\mathcal{U}(t_1, t_0) \equiv e^{\frac{i}{\hbar}H_{\mathrm{L}}t_1}U(t_1, t_0)e^{-\frac{i}{\hbar}H_{\mathrm{L}}t_0} \tag{76}$$

we can write

$$|\psi_{\mathrm{out}}\rangle = \mathcal{U}(t_1, t_0)\,|\psi_{\mathrm{in}}\rangle\,. \tag{77}$$

Taking the limit as $t \to \pm\infty$, we have $\mathcal{U}(\infty, -\infty) = \lim_{t_1\to\infty,\, t_0\to-\infty}\mathcal{U}(t_1, t_0)$ and so

$$|\psi_{\mathrm{out}}\rangle = \mathcal{U}(\infty, -\infty)\,|\psi_{\mathrm{in}}\rangle\,. \tag{78}$$

Choosing a restricted time range $(t, -\infty)$, i.e. starting at $t = -\infty$ and ending at some arbitrary $t > -\infty$, we define

$$\overline{\mathcal{U}}(t) \equiv \mathcal{U}(t, -\infty). \tag{79}$$

Thus, we write the output and input kets as

$$|\psi_{\mathrm{out}}\rangle = \left|\overline{\psi}(\infty)\right\rangle \tag{80}$$

and

$$|\psi_{\mathrm{in}}\rangle = \left|\overline{\psi}(-\infty)\right\rangle \tag{81}$$

respectively, where $\left|\overline{\psi}(t)\right\rangle = \overline{U}(t)\,|\psi_{\mathrm{in}}\rangle$. We can differentiate across this equation to obtain the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\overline{\psi}(t)\right\rangle = \overline{H}_{\mathrm{NL}}(t)\left|\overline{\psi}(t)\right\rangle \tag{82}$$

where

$$\overline{H}_{\mathrm{NL}}(t) \equiv e^{iH_{\mathrm{L}}t/\hbar}H_{\mathrm{NL}}e^{-iH_{\mathrm{L}}t/\hbar} \tag{83}$$

where t ranges from $-\infty$ to ∞, and as it does so, the ket goes from $|\psi_{\mathrm{in}}\rangle$ to $|\psi_{\mathrm{out}}\rangle$.

6 Numerical simulation

In this section, we write the simulation of this system for variable input pump power and duration, and describe the behaviour of the squeezing matrix with respect to these parameters. We then focus on the output photon statistics $n_{\mathrm{ph},S}$ and K – the output state's photon number and Schmidt number respectively. We generate plots showing the behaviour of these statistics for the variable input parameters over a defined experimentally reasonable range, and provide an elementary optimization to find the optimum input parameters to maximise $n_{\mathrm{ph},S}$ and minimise K, which is desirable in order to produce a single Schmidt mode squeezed state.

6.1 Building and extracting the squeezing matrix

Simulating this system requires us to define a few functions that do the heavy lifting for us. The first of these is a function that takes in the auxiliary matrices V and W and computes the squeezing parameter, squeezing phase, and squeezing matrix. The theory outlining the derivation of the auxiliary matrices for this system (and other systems, as the method is quite general) is given in Appendix B of [HSS] but for the author's own sanity, it is summarized here.

6.1.1 Equations for Gaussian parameters

Consider an input ket that is a coherent state in the actual input channel and a vacuum in all others:

$$|\psi_{\text{in}}\rangle = D_b(\boldsymbol{\beta}_{\text{in}})\,|\text{vac}\rangle \tag{84}$$

where $D_b(\boldsymbol{\beta}_{\text{in}})$ is the (unitary) displacement operator (for all the non-input channels, labelled by b) with displacement parameter $\boldsymbol{\beta}_{\text{in}} \equiv \boldsymbol{\beta}(-\infty)$. Let the vectors of the bosonic creation and annihilation operators for this channel be $\mathbf{b}^\dagger$ and $\mathbf{b}$ respectively:

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}, \quad \mathbf{b}^\dagger = \begin{bmatrix} b_1^\dagger \\ b_2^\dagger \\ \vdots \end{bmatrix}. \tag{85}$$

We seek a solution to the Schrödinger equation of the form

$$\left|\overline{\psi}(t)\right\rangle = U_a(t)U_b(t)\left|\tilde{\psi}\right\rangle \tag{86}$$

where $\left|\tilde{\psi}\right\rangle$ encapsulates all the non-Gaussian behaviour of the full solution ket; in effect, isolating it. The two unitary evolution operators are

$$U_a(t) = S_a(\mathbf{J}(t))R_a(\boldsymbol{\phi}(t))e^{i\theta_a(t)}, \quad U_b(t) = D_a(\boldsymbol{\beta}(t))e^{i\theta_b(t)} \tag{87}$$

and their product is simply the operator product of the squeezing, rotation, and displacement operators together with the a global phase term. This follows directly from [MaRhodes], which shows that the multimode unitary time evolution operator associated with a multimode quadratic Hamiltonian can always be represented as this operator product, and such a representation is unique. Each of the Gaussian parameters $\mathbf{J}$, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$ is subject to certain initial conditions, and we take the initial ket $\left|\tilde{\psi}(-\infty)\right\rangle$ to be the vacuum. We then introduce an effective Hamiltonian H_{eff} comprised of Hamiltonians associated with U_a (H_a) and U_b (H_b). The former describes the generation of photon pairs by the SFWM process while the latter describes the rate of pump amplitude (the $\boldsymbol{\beta}(t)$ parameter) depletion. Requiring that H_{eff} equals the Hamiltonian $H_{\text{I}}(t)$ which contains only non-Gaussian terms and is comprised of the auxiliary matrices $\mathbf{V}(t)$ and $\mathbf{W}(t)$ necessitates a choice of time-dependence of the Gaussian parameters that makes this happen – it turns out that the condition is satisfied by choosing $\mathbf{V}(t)$, $\mathbf{W}(t)$, $\boldsymbol{\beta}(t)$, and $\theta_b(t)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}(t) = -2i\boldsymbol{\zeta}(t)\mathbf{W}^*(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{W}(t) = -2i\boldsymbol{\zeta}(t)\mathbf{V}^*(t), \quad i\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\beta}(t)]_l = [\boldsymbol{\gamma}(t)]_l \quad \text{and}$$

$$\theta_b(t) = \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t'\,[\boldsymbol{\gamma}(t')]_l[\boldsymbol{\beta}^*(t')]_l + \text{c.c.} \tag{88}$$

The zeta matrix was introduced earlier, and the vector $\boldsymbol{\gamma}(t)$ determines the rate of pump depletion. The extraction of the parameters $\mathbf{J}(t)$ and $\boldsymbol{\phi}(t)$ from $\mathbf{V}(t)$ and $\mathbf{W}(t)$ is explicitly shown in [HSS Appendix C]. This is implemented in our Julia simulation by the `extract_J` function, which takes in the auxiliary matrices and returns the squeezing matrix, squeezing parameter, and squeezing phase. It is reproduced in [Appendix ?].

6.2 Building the nonlinear coefficient

We construct the Λ tensor as a function of the pump photon numbers nph_P, nph_C, the pump powers U_P, U_C, the pump durations τ_P, τ_C, and the dimensionless time parameter $\bar{t}$. We

write the prefactor, define the initial arrays for Λ and Λ_0 (four-dimensional, taking complex float inputs) and reshape the dimensionless x_i parameters for broadcasting, which helps us avoid four nested loops, saving us a large amount of time. We compute the phase term and denominator terms, and bring them together in Λ_0 by multiplying the denominator term by the exponential of the phase term. Λ is then found by multiplying Λ_0 by the appropriate terms in η_S (the squeezed light coupling efficiency to the actual channel).

6.3 Solving the ODEs and defining the system parameters

The coupled ODEs for $\mathbf{V}$ and $\mathbf{W}$ are solved using a function (`solve_ode`) we write that takes the time span, pulse shapes, pulse photon numbers, pulse powers, and pulse durations and returns the solution to the ODE problem we define using another function (`ode!`) nested within. The nested function defines the vectors of the auxiliary matrices as sections (specifically, column vectors) of a larger matrix and defines the matrices themselves as reshapes of the vectors as n by n matrices. Using the nonlinear coefficient function, we solve for the $\boldsymbol{\zeta}$ matrix by writing a loop with respect to the two indices of the matrix and defining each element as

$$[\boldsymbol{\zeta}]_{ij} = \boldsymbol{\beta}_P^{\intercal} \Lambda_{ijkl} \boldsymbol{\beta}_C \tag{89}$$

The differential elements of $\mathbf{V}$ and $\mathbf{W}$ are defined as

$$\mathrm{d}\mathbf{V} = -\frac{i}{\overline{\overline{\Gamma}}_S}(\boldsymbol{\zeta} + \boldsymbol{\zeta}^{\intercal})\mathbf{W}^* \tag{90}$$

$$\mathrm{d}\mathbf{W} = -\frac{i}{\overline{\overline{\Gamma}}_S}(\boldsymbol{\zeta} + \boldsymbol{\zeta}^{\intercal})\mathbf{V}^* \tag{91}$$

The ODE system is then set up with the inbuilt ODEProblem function and solved with the Tsit5 solver with tolerances of 1×10^{-7} and a maximum number of iterations 1×10^6.

We then define all the system parameters and constants with their units. They are listed with their values in the conjoining code appendices.

6.4 The `mapmaker` function

We define a function that takes in all of the arguments that the aforementioned extraction, nonlinear coefficient, and solve ODE functions feature together with the pulse shapes β_P and β_C which are now explicitly defined, and returns two photon statistics: the total number of generated photons in the squeezed state $n_{\mathrm{ph}S}$ and the Schmidt number of the squeezed state K. It does this by solving the coupled ODE system for given parameter combinations of input pulse energies U_P, U_C and durations τ_P, τ_C, building the squeezing matrix, and extracting the squeezing parameters. The total number of generated photons is given by

$$n_{\mathrm{ph}S} = \mathrm{Re}(\mathrm{Tr}[\mathbf{W}^{\dagger}\mathbf{W}]) \tag{92}$$

and the Schmidt number is given by

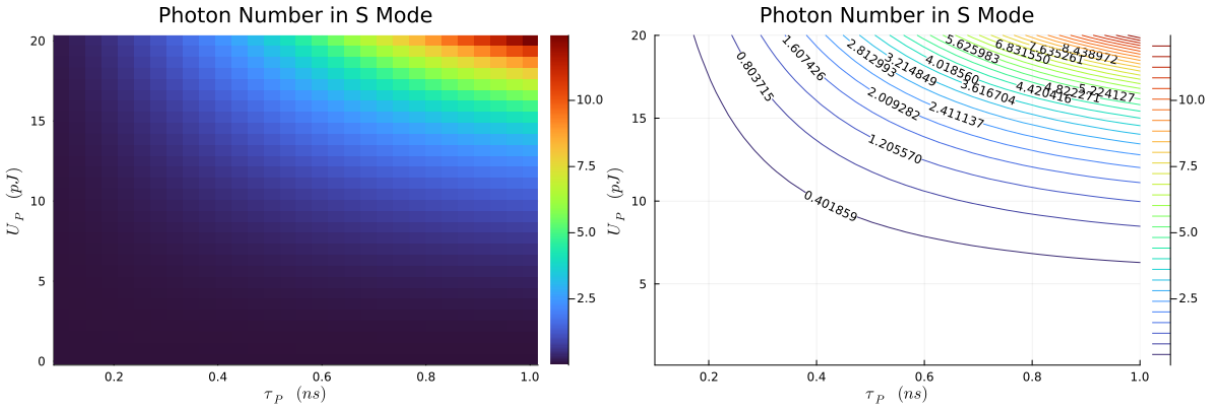
$$K = \frac{(n_{\mathrm{ph}S})^2}{\sum (\sinh(r))^4} \tag{93}$$

We define ranges of parameter values: τ_P and τ_C go from 0.5 ns to 3.0 ns with an interval length of l, while U_P and U_C go from 10 pJ to 50 pJ with an interval length of l. The variable interval length is equivalent to the resolution of the resulting plots for $n_{\mathrm{ph}S}$ and K. We then write a simple loop that runs over the indices for the duration and energy ranges respectively, solving the ODE system for each parameter combination $(\tau_P, \tau_C, U_P, U_C)$. We then show the results

for each photon statistic as heatmaps with overlaid contour plots, showing the isoclines for the statistics with respect to the parameter combinations given. As U_P and U_C take on the same values in the defined ranges (and similarly for τ_P and τ_C), we simply label the axes with the P mode durations and energies with the values for the C mode energies and durations being implicitly understood as being the same.

6.5 Heatmaps and contour plots

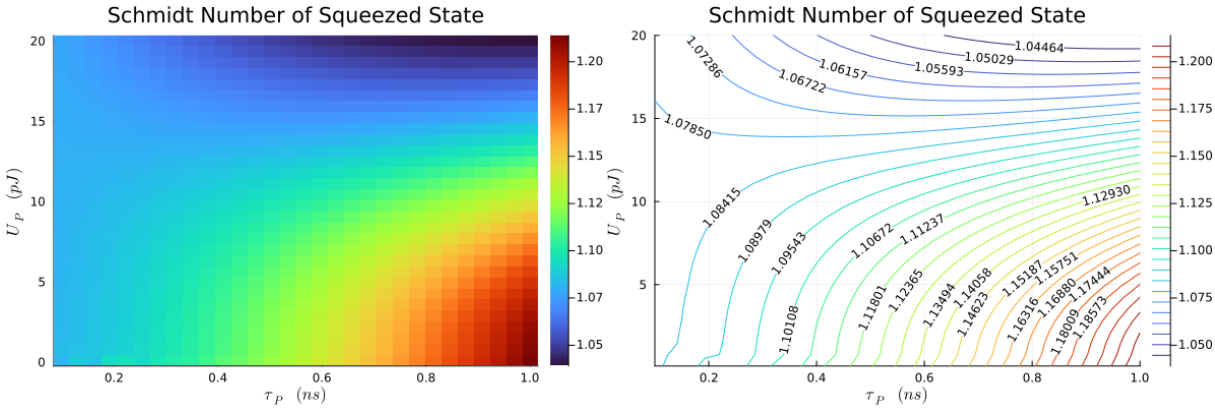
We show the resulting plots for a resolution $l \times l = 30 \times 30$ and ranges $\tau_P, \tau_C = [0.0\text{ ns}, 1.0\text{ ns}]$, $U_P, U_C = [0.0\text{ pJ}, 20.0\text{ pJ}]$.



(a) Heatmap for $n_{\mathrm{ph}S}$.

(b) Contour map for $n_{\mathrm{ph}S}$.

Figure 1: Maps for $n_{\mathrm{ph}S}$ produced by `matrixmaker`.



(a) Heatmap for K.

(b) Contour map for K.

Figure 2: Maps for K produced by `matrixmaker`.

Note that these are preliminary plots; the behaviour for $\tau_P < 0.5$ ns is unreliable and not to be trusted!

Interpretations: