The iron curtain of gravitational collapse

A review of the cosmic censorship conjectures

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1 Introduction

Stephen Hawking and Roger Penrose presented the current picture of gravitational collapse in their seminal 1970 paper [HP70]. They formulated that for bodies with too much mass concentrated within too small a volume, there is a collapse process that is unstoppable, eventually leading to the creation of a singularity in the structure of spacetime. Physically, the idea of such a singularity strictly occurring in nature is predicated on the assumption that the laws of quantum mechanics do not intervene in a change of the structure of spacetime that the general theory of relativity (GR) classically describes. In our context, a singularity refers to a region where our conventional classical description of spacetime is no longer applicable.[1] The standard picture of collapse to a black hole [Pen69] says that the singularity is not visible to distant observers because it is hidden, in a manner of speaking, by an absolute event horizon. This means that whatever physics happens at the singularity is not accessible by such an observer. This is the 'censorship' we are so interested in.

2 Preliminary concepts

We first understand where we are working and what we are working with. Recall that a spacetime $(\mathcal{M}, g)$ is a $(3+1)$-dimensional manifold equipped with the Lorentzian metric g, which is a symmetric and nondegenerate bilinear form defined on $V_p \times V_p$ where V_p is the tangent space to the manifold at a point $p \in \mathcal{M}$. The signature of the metric is $(-,+,+,+)$. The most basic example is that of Minkowski spacetime $(\mathbb{R}^{3+1}, \mu)$, equipped with the metric

$$\mu = \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{1}$$

The metric of a spacetime defines a light cone on the tangent space of each point (i.e. an event) on the manifold; a vector belonging to this tangent space can be classified as follows [Wal10]:

Definition 1 (Timelike, lightlike, spacelike)**.** *A vector X in the tangent space of a point on (M, g) is classified as*

- *Timelike:* $g(X, X) < 0$.
- *Lightlike/null:* $g(X, X) = 0$
- *Spacelike:* $g(X, X) > 0$

Observers in free-fall follow timelike or null geodesics, since the upper bound on the speed of information travel is the speed of light.

[1] We would normally expect this to happen only when we're dealing with radii of curvature on the scale of the Planck length.

2.1 Some essential definitions

The following definitions are due to Carroll [Car19]:

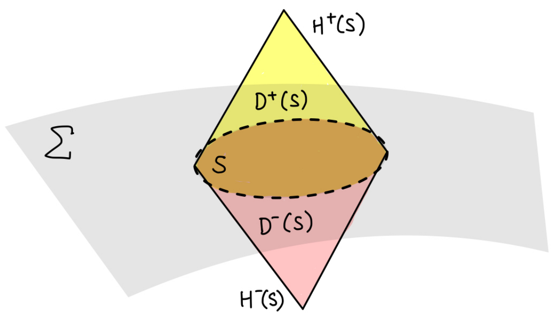
Definition 2 (Causal curves)**.** *A curve (not necessarily a geodesic) that is either timelike or null is called a causal curve.*

Definition 3 (Causal futures/pasts and chronological futures/pasts)**.** *Let S be a subset of $\mathcal{M}$. The causal future of S (denoted $J^+(S)$) is the set of points reachable from S via a future-directed causal curve. The chronological future $I^+(S)$ is a restriction of this; it is the set of points reachable by a future-directed timelike curve. The causal past J^- and chronological past I^- are similarly defined, but for past-directed causal and timelike curves respectively.*

Definition 4 (Achronal sets)**.** *$S \subset \mathcal{M}$ is achronal if there do not exist any two points p, q in S such that p and q can be connected by some timelike curve.*

Definition 5 (Future/past domain of dependence)**.** *Let S be a closed achronal set. The future domain of dependence of S, denoted $D^+(S)$, is the set of all points p such that every past-moving inextendible (i.e. not terminating at some finite point) causal curve through p must intersect S. The past domain of dependence $D^-(S)$ is defined similarly. The domain of dependence is simply $D(S) = D^+(S) \cup D^-(S)$.*

Definition 6 (Cauchy horizons)**.** *The boundary of $D^+(S)$ is the future Cauchy horizon $H^+(S)$. The boundary of $D^-(S)$ is the past Cauchy horizon $H^-(S)$. They are both null surfaces.*



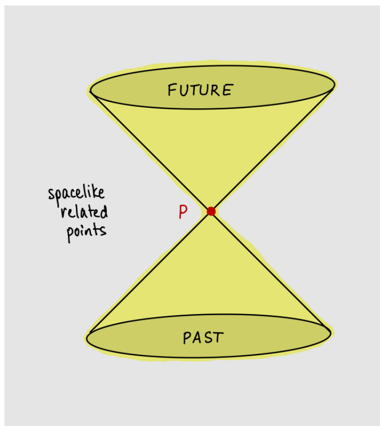


Figure 1: Left: The domains of dependence and Cauchy horizons for some achronal $S \subset \Sigma$. Right: The light cone of a point in $(\mathcal{M}, g)$.

Definition 7 (Cauchy surface)**.** *A closed achronal surface Σ is a Cauchy surface if the domain of dependence $D^+(\Sigma)$ is the entire manifold. Given information on a Cauchy surface, we can predict what happens throughout the entire spacetime. If a spacetime has a Cauchy surface, then it is called globally hyperbolic.*

With these definitions, we now give the formulation of the cosmic censorship conjectures.

3 Formulations

Before we dare to formulate the cosmic censorship conjectures as lone-standing statements, we first examine the processes leading to the formation of the singularities that the conjectures are concerned with.

3.1 Gravitational Collapse

The classical picture of collapse is centred around Schwarzschild's solutions to Einstein's equations in a vacuum setting. In the coordinates used by Schwarzschild, the metric is of the form

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\;\mathrm{d}\phi^2) \qquad \text{with } c = G = 1 \tag{2}$$

where θ and ϕ are the usual spherical polar angle coordinates. We chose the radial coordinate r such that each sphere defined by r constant and t constant has intrinsic[2] surface area $4\pi r^2$. The metric form is invariant under the transformations $t \to t + \mathrm{C}$ where C is some constant and $t \to -t$. The mass of the body collapsing is m. From (2), it is evident that at $r = 2m$, the form of the metric breaks. This radius is called the body's Schwarzschild radius.
Consider the situation of a spherically symmetric, nonrotating star undergoing collapse (figure 2). While the star's spherical symmetry remains, the applicable metric is (2). Points at the star's surface are described by timelike lines and since the light cones of these points seem to become narrower as they approach the Schwarzschild radius, one might think that this radius where a singularity is formed. However, a freely falling observer (following a timelike/null geodesic) will find that the elapsed total proper time they measure is finite. After this proper time has elapsed, there are two possible scenarios:

1. A spacetime singularity lies waiting, and the observer is torn apart and sent into oblivion.
2. A new world awaits that requires some other coordinate system to be described.

Scenario 2 is what the observer will encounter. The 'singularity' at $r = 2m$ is not a true singularity; it is simply an artifact of the coordinates that the Schwarzschild metric is written in. We can replace the time coordinate in (t, r, θ, ϕ) by

$$v = t + r + 2m\log(r - 2m). \tag{3}$$

In the new coordinates (v, r, θ, ϕ) (known as Eddington-Finkelstein coordinates), the metric (2) becomes

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}v^2 - 2\;\mathrm{d}r\;\mathrm{d}v - r^2(\mathrm{d}\theta^2 + \sin^2\theta\;\mathrm{d}\phi^2) \tag{4}$$

This metric is still valid at $r = 2m$. In fact, all of $r \in (0, \infty)$ is covered by this metric and the region $r > 2m$ is equivalently described by (2) and (4). The region $0 < r < 2m$ is also described perfectly well; the only radius where the Schwarzschild metric in Eddington-Finkelstein coordinates breaks down is at $r = 0$. This is where we find the true singularity. In the left figure in figure (2), the observer travelling along a timelike/null geodesic will see that there is nothing special about $r = 2m$ and that $r = 0$ is where they will certainly encounter scenario 1 from the possible scenarios mentioned earlier. The increasingly warped curvature of spacetime is physically felt in the form of immense tidal forces, resulting in what we call 'spaghettification'. The Schwarzschild radius $r = 2m$ is referred to as the ' absolute event horizon', since it is the strict boundary of the set of events that can be observed by an external inertial observer (in principle).

[2] i.e. can be measured within the surface itself with no reference to some larger space containing it.

This classical picture of spherically symmetric gravitational collapse has raised a number of possible objections and provoked inquiries about the nature of the event horizon and the physics beyond it. However, it remains a good foundation for further discussion on cosmic censorship.

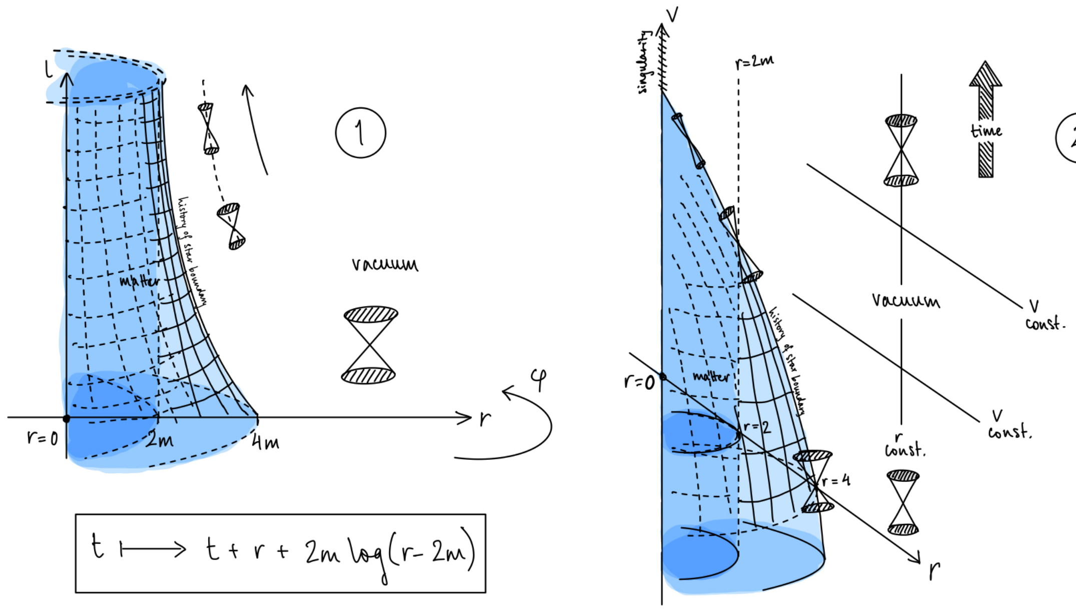


Figure 2: Spherically symmetric gravitational collapse in (1) Schwarzschild coordinates and (2) Eddington-Finkelstein coordinates. Reproduced from [Penrose1999].

3.2 Initial view of cosmic censorship

Initially, the picture of censorship can be given as follows:

In a generic gravitational collapse, the resulting spacetime singularity is shielded from view by an absolute event horizon.

From this initial picture, we would say that the alternative to a hidden singularity, a visible singularity, would not occur generically (i.e. in your typical astrophysical situations) – that is, there may exist exceptions where we have some magnificently special collapse. Satisfying a standard criterion for unstoppable gravitational collapse simply requires a sufficiently large mass to become confined into a sufficiently small region. For example, in the circumstance of a large galaxy, there is no any reason to expect that there would be some physics that takes precedence in order to prevent unstoppable collapse, but we cannot immediately infer that a black hole will be produced in such a situation. For this, we must assume that in GR, this notion of cosmic censorship holds true in some way.

Remark (Mathematical criteria for unstoppable collapse)**.** *These are the criteria:*

1. *The existence of a trapped surface.*

2. *The existence of a point whose future light cone begins to reconverge in every direction along the cone.*

A trapped surface is defined as follows (due to [Pen65]):

Definition 8 (Trapped surface)**.** *A trapped surface is generally defined to be a closed, spacelike 2-surface T^2 with the property that the two systems of null geodesics which meet T^2 orthogonally converge locally in future directions at the surface.*

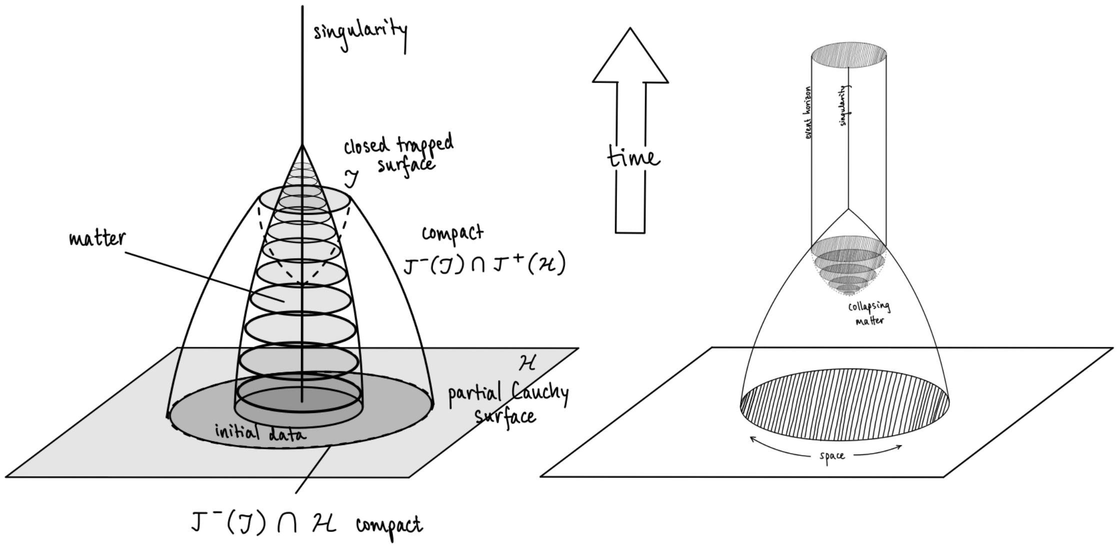


Figure 3: The gravitational collapse of a nonrotating spherical star leading to a singularity, with one spatial dimension suppressed. Reproduced from Fig 55 in [HE23].

Now, in either situation, (criterion 1 or criterion 2), with some physically reasonable assumptions: (1) nonnegativity of energy, (2) nonexistence of closed timelike curves, (3) some genericity condition (eg. assuming every causal geodesic contains at least one point at which the Riemann curvature is is not lined up in a particular way with the geodesic, i.e. with a tangent vector ξ^a, there is at least one point where $\xi_{[a}R_{b]cd[e}\xi_{f]}\xi^c\xi^d = 0$), then by [HP70] a spacetime singularity of some kind must occur.

Deducing that a black hole type singularity must occur whenever a trapped surface is formed is not as direct as it might seem; you need the assumption of cosmic censorship for this deduction to be valid. Even worse, deducing that any sort of spacetime singularity occurs at all requires us to assume something nontrivial, eg. the very existence of a trapped surface in the first place. So, there being a trapped surface does not imply the nonexistence of a singularity, and even more so, there *not* being a trapped surface does not tell us anything about there *being* a singularity.

3.3 Old formulations

As seen with the classical collapse picture described in the previous section, the prototypical singular spacetime (i.e. a spacetime featuring singularities. Note that not all curved spacetimes in GR are singular!) is the Schwarzschild spacetime. The Penrose diagram in figure 3 describes the causal structure of the maximal analytic[3] Schwarzschild singularity. The first Penrose diagram is for the maximally analytic extended Schwarzschild spacetime, with a spacelike singularity

[3] i.e. extended in terms of Eddington-Finkelstein coordinates beyond the Schwarzschild coordinate-induced singularity at $r = 2m$.

$\mathcal{S} = \{r = 0\}$ hidden behind an absolute event horizon $\mathcal{H}^+$, i.e. $\mathcal{S} \cap \mathcal{J}^-(\mathcal{I}^+) = \varnothing$. Here, $\mathcal{J}^\pm$ are the causal future/past of the null infinity lines $\mathcal{I}^\pm$. These are regions of 'conformal infinity', i.e. infinity in the sense of conformal diagrams. The second Penrose diagram for negative-mass Schwarzschild spacetime. Again, $\mathcal{S} = \{r = 0\}$ is a true, i.e. curvature singularity but it is timelike and $\mathcal{S} \subset \mathcal{J}^-(\mathcal{I}^+)$. Hence, the singularity is not hidden and is 'naked'. The third Penrose diagram models the deep interior of Reissner-Nordström and Kerr black holes. Here, the singularity is timelike and can be thought of as 'locally naked', i.e. it is not visible at infinity but in the neighbourhood of the singularity, it is no longer hidden.

These examples of singularities give us some intuition about the singularities mentioned in the weak and strong cosmic censorship conjectures.

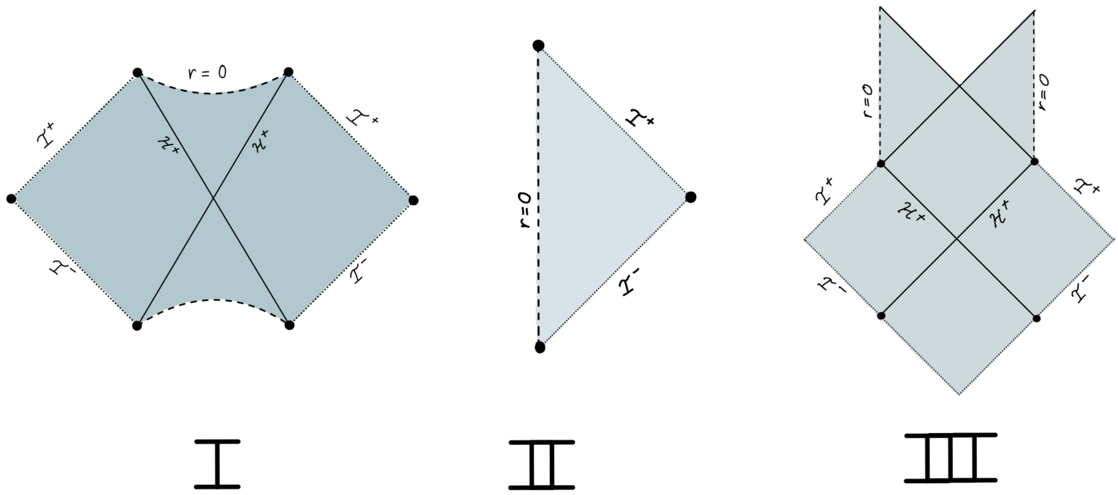


Figure 4: Penrose diagrams for (I) maximal analytic Schwarzschild spacetime (II) negative-mass Schwarzschild (III) Reissner-Nordström and Kerr black holes.

Theorem 1 (Weak cosmic censorship. Penrose, 1969)**.** *In gravitational collapse, singularities are always cloaked by horizons, i.e.* $\mathcal{S} \cap \mathcal{J}^-(\mathcal{I}^+) = \varnothing$*.*

Theorem 2 (Strong cosmic censorship, Penrose 1972)**.** *In generic gravitational collapse, there are no 'locally naked' singularities, i.e. singularities are generically spacelike or null, and not timelike.*

Remark. *In Figure 3.II, we showed the situation of a negative-mass Schwarzschild, which is unphysical; hence, it is not considered to be a generic situation. There is no gravitational collapse from a generic initial astrophysical situation; the naked singularity is always present. Hence, it does not contradict the statement of weak cosmic censorship. The third Penrose diagram (that of the Reissner-Nordström timelike singularity) is a physical situation, but the presence of the genericity condition in this formulation of the strong cosmic censorship conjecture is essential to making sure that Figure 3.III does not violate it.*

There is an even stronger statement, due to Mihalis Dafermos, that he calls the 'very strong cosmic censorship conjecture':

Theorem 3 (Very strong cosmic censorship)**.** *In gravitational collapse, singularities are generically spacelike.*

3.4 Modern formulations

3.4.1 The Cauchy problem

Before the modern formulations of the conjectures are shown, we introduce the Cauchy problem. Yvonne Choquet-Bruhat's seminal 1952 paper showed that Einstein's equations can be constructed as an initial value (Cauchy) problem, i.e. given initial data, there exist corresponding solutions to the Einstein equations. This result is extremely powerful; it shows that GR is a predictive, deterministic theory. To understand the central theorem by Choquet-Bruhat, we provide a few definitions from [CG69]:

Definition 9 (Extrinsic curvature)**.** *Let S be a smooth spacelike 3-surface in a spacetime M. For each $p \in S$, we may draw the unique timelike geodesic orthogonal to S. The field ξ^a of future-directed unit tangent vectors to this geodesic satisfies the equations*

$$\nabla_a \xi_b = \nabla_{[a}\xi_{b]}, \quad \xi^a \nabla_a \xi_b = 0 \tag{5}$$

in a neighbourhood of S. Then, the tensor field $\nabla_a \xi_b$ defined in a neighbourhood of S induces the symmetric tensor field $C_{\alpha\beta}$ on S. In other words, the extrinsic curvature is the rate of change of the unit tangent ξ^a with respect to S, where ξ^a is tangent to the geodesics normal to S.

Definition 10 (Initial data set)**.** *An initial data set is a smooth 3-dimensional manifold Σ endowed with a positive-definite metric $h_{\alpha\beta}$ and a symmetric tensor field (the extrinsic curvature) $C_{\alpha\beta}$ subject to the constraint equations*

$$R + C^\alpha_\beta C^\beta_\alpha - (C^\alpha_\alpha)^2 = 0, \quad \nabla_\alpha(C^\alpha_\beta - \delta^\alpha_\beta C^\gamma_\gamma) = 0 \tag{6}$$

where R is the curvature scalar of Σ.

Definition 11 (Development of initial data)**.** *A development of an initial data set is a spacetime $\mathcal{M}$ (a solution of Einstein's equations) along with a diffeomorphism of Σ onto a spacelike 3-dimensional submanifold of $\mathcal{M}$, such that*

1. *The metric and extrinsic curvature of S (inherited from $\mathcal{M}$) coincide with the images of $h_{\alpha\beta}$ and $C_{\alpha\beta}$ given as initial data on Σ.*
2. *S is a Cauchy surface for $\mathcal{M}$.*

We may identify S with Σ because the former is the image of the latter.

Thus, the local existence theorem can be stated as follows:

Theorem 4 (Local existence for the Cauchy problem)**.** *Every initial data set has a development.*

Definition 12 (Extension)**.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be developments of the initial data set S. Then, $\mathcal{M}$ is an extension of $\mathcal{M}'$ if there exists an isometry from $\mathcal{M}'$ onto some $T \subset \mathcal{M}$ such that every point of S is invariant under the isometry. Then, we can regard $\mathcal{M}'$ to be embedded in $\mathcal{M}$. See Figure 5.*

Consider the following theorem due to Choquet-Bruhat and later developed further with Robert Geroch in [CG69]:

Theorem 5. *Let $(\Sigma, \bar{g}, K)$ be a smooth vacuum initial data set (comprising the smooth hypersurface Σ, the metric $\bar{g}$ on Σ, and the extrinsic curvature K on Σ). Then, there exists a unique smooth spacetime $(\mathcal{M}, g)$ such that*

1. *$Ric(g) = 0$.*

2. *$(\mathcal{M}, g)$ is globally hyperbolic with Cauchy surface Σ with induced first and second fundamental forms $\bar{g}$, and K respectively.*

3. *Any other smooth spacetime with properties 1 and 2 isometrically embeds into $\mathcal{M}$.*

$(\mathcal{M}, g)$ is called the maximal Cauchy development, i.e. it is the largest globally hyperbolic spacetime that evolves from the initial data in the sense of development.

For the proof of this theorem, we refer the reader to [CG69]. While this specific version is for an initial data set in vacuo, theorems along similar lines can be proven for Einstein-dust fields, Einstein-matter fields, and the like.

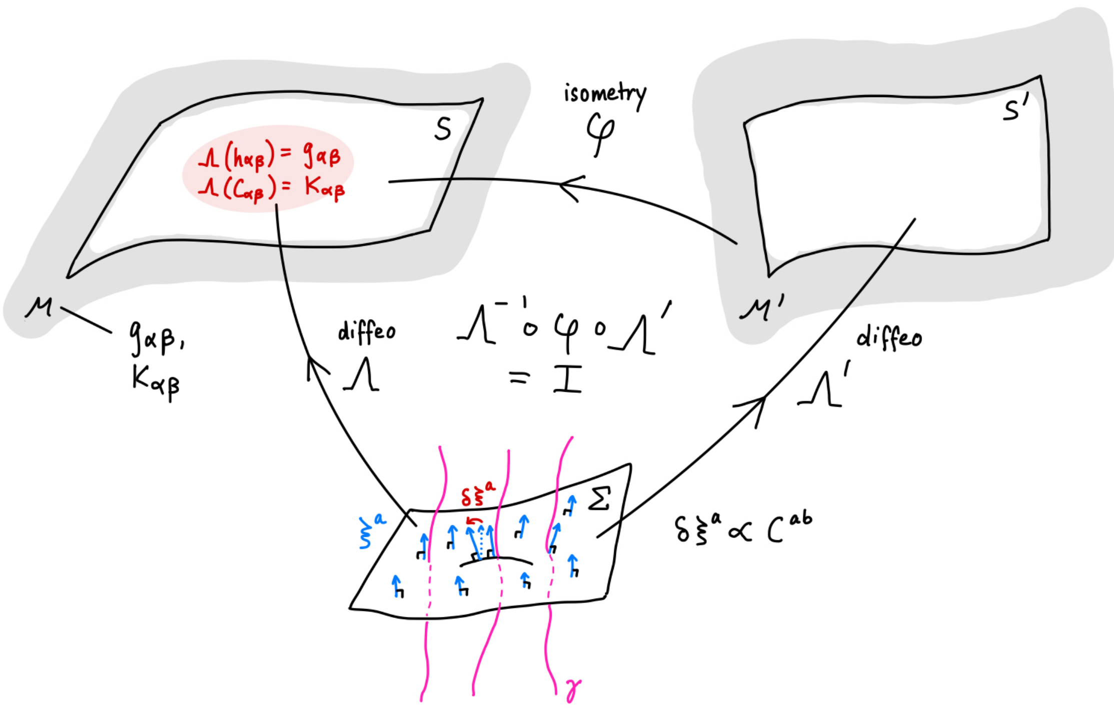


Figure 5: $\mathcal{M}$ is an extension of $\mathcal{M}'$. If Λ and Λ' are diffeomorphisms from Σ into subsets S and S' of $\mathcal{M}$ and $\mathcal{M}'$ respectively, and if φ is the isometry from $\mathcal{M}'$ onto $S \subset \mathcal{M}$ then $\Lambda^{-1}\varphi\Lambda'$ is the identity mapping $I : \Sigma \to \Sigma$.

We must accept this theorem in order to do any further work; if we do not assume it to hold, then the theory is no longer deterministic and we cannot make predictions. Accepting the primacy of this theorem means that assumptions may only be made[4] on the initial data Σ and whatever properties we wish to look at must be describable in terms of maximal Cauchy developments $(\mathcal{M}, g)$. Since the maximal Cauchy development $(\mathcal{M}, g)$ is globally hyperbolic (that is, it has a Cauchy surface), the finite boundary of $\mathcal{M}$ cannot be timelike anywhere. Thus, by definition, there is no such thing as a timelike singularity. Hence, it makes little sense to say that cosmic censorship is a statement about the fact that there cannot be timelike singularities (as done in the 'old' formulation of the strong cosmic censorship conjecture), since that is trivially true – hence the reason for a modern formulation of cosmic censorship.

Denoting the future null infinity of $\mathcal{M}$ as $\mathcal{I}^+$, we necessarily have $S \setminus \mathcal{J}^-(\mathcal{I}^+) = \varnothing$.

[4] i.e. we cannot make any assumptions (genericity, etc.) on the spacetime constituting the maximal development since it is solely a product of the initial data.

3.4.2 Schwarzschild in the context of Cauchy evolution

We return to our maximally analytic Schwarzschild singularity Penrose diagram and try to apply the notions we have developed to see what Cauchy evolution looks like in action.

Consider a set of initial data that is 2-ended and asymptotically flat on the hypersurface Σ, as visualised in Figure 6. The future Cauchy development is maximal but $(\mathcal{M}, g)$ is geodesically incomplete; that is, it has geodesics of finite affine length.[5] This incompleteness is stable with respect to a perturbation of the initial data on Σ. In conjunction with the singularity theorem of [Pen65], this means that under a perturbation of the initial data, the resulting maximal development $(\mathcal{M}, g)$ will still possess a trapped surface. Taking this and the global hyperbolicity of $\mathcal{M}$ into account, we see that geodesic incompleteness is preserved. However, independent of the fact that $\mathcal{M}$ is geodesically incomplete, the future null infinity[6] $\mathcal{I}^+$ is complete [GH79]. $\mathcal{M}$ is also not C^2, since it contains the curvature singularity $r = 0$ but [Sbi15] showed that the manifold is inextensible as a continuous Lorentzian manifold, resulting in the physical effect of crushing tidal forces on observers approaching $r = 0$. The singularity itself can be characterised as a spacelike boundary $\mathcal{S} = \{r = 0\}$ of $\mathcal{M}$.

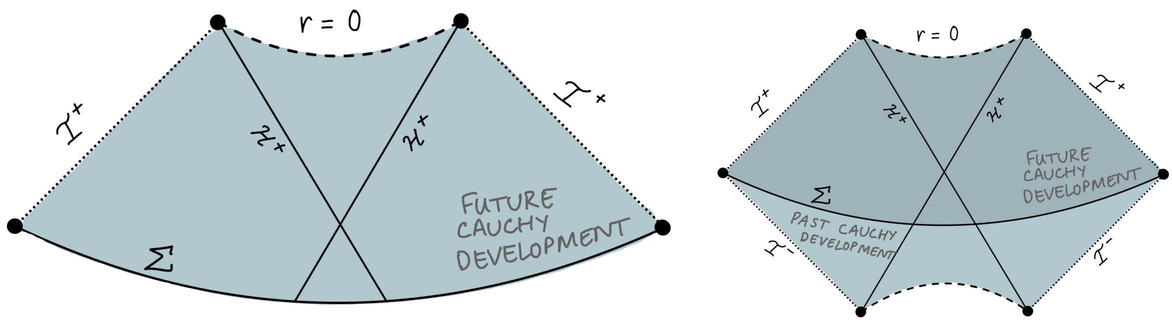


Figure 6: Σ is a 2-ended, asymptotically flat hypersurface. Strictly speaking, this is not a physical set of initial data but it is okay for our illustrative purposes.

Figure 7 shows the case of the negative-mass Schwarzschild singularity. The maximal future Cauchy development of the hypersurface Σ is the darker shaded region (shown in isolation on the right). Recall that since we can only talk about the properties in terms of the maximal Cauchy development, we must find a way to characterise the 'naked' singularity in terms of this structure. This was done by [GH79]; the segment of future null infinity $\mathcal{I}^+$ is incomplete. In terms of the gravitational wave experiment description we gave for the Cauchy development for the usual Schwarzschild case, this would mean that there exists a time in the future where such experiments would have to cease in the context of measuring gravity waves from this naked singularity. The power of this formulation is that we no longer have to explicitly refer to the singularity; everything can be characterised in terms of the Cauchy problem.

[5] Wald states that a more precise characterisation of geodesic incompleteness is that "there exist geodesics that are inextendible in at least one direction but have only a finite range of affine parameter." He further states that a spacetime is defined to be singular if it possesses at least one incomplete geodesic.

[6] We might say that gravitational wave experiments are performed in this region

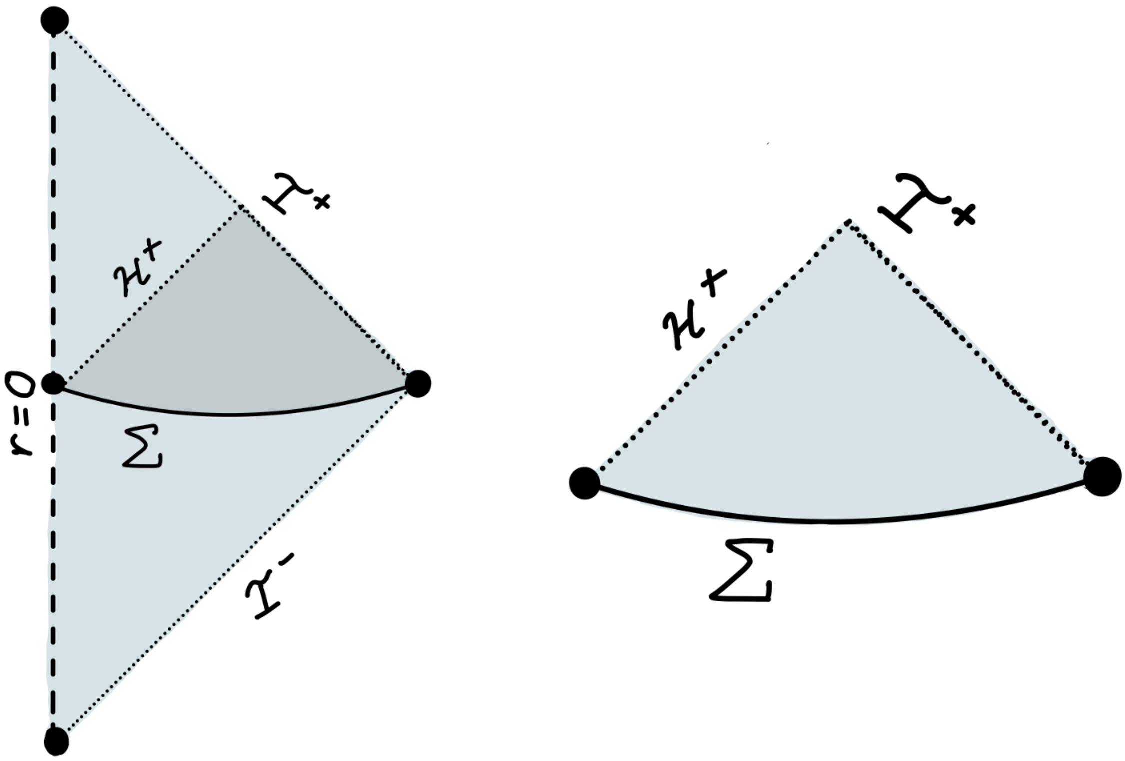


Figure 7: Σ is a hypersurface that is asymptotically flat towards the rightmost apex of the triangle but incomplete at $r = 0$ due to the presence of the singularity.

Note the presence of the Cauchy horizon $\mathcal{H}^+$ of $\mathcal{M}$.[7] We may smoothly extend the maximal Cauchy development $(\mathcal{M}, g)$ to a larger spacetime $(\tilde{\mathcal{M}}, g)$ across this horizon (rendering $\mathcal{H}^+$ a null hypersurface in these constructions) resulting in non-globally hyperbolic extensions that are not uniquely determined by the initial data, i.e. $\tilde{\mathcal{M}}$ is non-unique. Thus, we have spacetimes that no longer adhere to the Cauchy problem as formulated by Choquet-Bruhat. This is a failure of determinism; we cannot make predictions in these extensions. However, recall that in constructing the hypersurface as shown in Figure 7, we have made an arbitrary choice independent of the initial data. Σ is incomplete, so we can say that the spacetimes in the cases of negative-mass Schwarzschild and Reissner-Nordström/Kerr, containing timelike singularities, are not physically valid.

This idea can be extended to the case of the Reissner-Nordström/Kerr spacetime. In this case, however, the spacetimes can be extended in such a way that all incomplete geodesics can pass through to the spacetime extension, so they may be regarded as maximal Cauchy developments in and of themselves. There is nothing singular about this spacetime, since it does not possess a finite geodesic. With reference to the original formulations of the cosmic censorship conjectures, we can now think of them as statements about determinism, rather than just singularities.

3.5 Modern formulations

We now attempt to provide a more modern formulation of weak cosmic censorship given the machinery of the Cauchy problem we have acquired.

[7] [Hawking1967] provides one of the first views viz. Cauchy horizons with respect to the notion of Cauchy developments.

Conjecture 1 (Weak cosmic censorship, Christodolou 1999)**.** *For complete, asymptotically flat initial data in vacuo, the maximal Cauchy development has a complete null infinity $\mathcal{I}^+$. [Chr99]*

There is no mention of singularities or the like; this is a global existence statement that is perfectly compatible[8] with Penrose and Hawking's original singularity theorems.

Conjecture 2 (Strong cosmic censorship, Christodolou 1999)**.** *For generic, complete, asymptotically flat initial data in vacuo, the maximal Cauchy development is future inextendible as a suitably regular Lorentzian manifold. [Chr99]*

This is a statement of global uniqueness, equivalent to determinism. There is no sense of this statement being 'stronger' than the weak cosmic censorship statement; together, they are statements of global existence and uniqueness. Building from these statements, Dafermos's 'very strong' cosmic censorship is as follows:

Conjecture 3 (Very strong cosmic censorship, Dafermos)**.** *For generic, asymptotically flat initial data in vacuo, the maximal Cauchy development is future inextendible as a Lorentzian manifold with metric assumed to be merely continuous. Moreover, the finite boundary of spacetime is spacelike.*

External inertial observers travelling along incomplete geodesics will find themselves in the presence of infinite curvature generating destructive tidal forces; informally, they will be sent into oblivion by infinite tidal deformations.

3.5.1 Amending weak cosmic censorship – spherical symmetry consideration

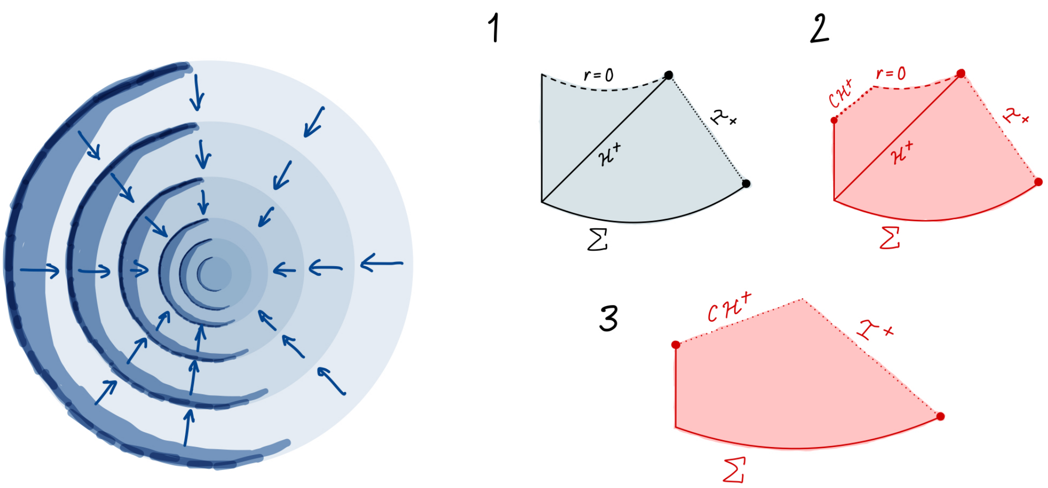


Figure 8: Left: A homogeneous dust ball undergoing gravitational collapse. Right: (1) Penrose diagram for the collapse of the dust ball as modelled by Oppenheimer-Snyder. (2), (3) Penrose diagrams due to Christodolou, modelling Cauchy developments for examples of initial data.

Consider a homogeneous dust ball exhibiting spherical symmetry undergoing gravitational collapse as modelled by Oppenheimer and Snyder in [OS39]. There is a singularity $\mathcal{S} = \{r = 0\}$, a acuum initial data set on the hypersurface Σ that is complete and asymptotically flat.

[8]However, we will see that this statement does not hold in general; under an Einstein scalar field system exhibiting spherical symmetry, it fails. To make it globally applicable, we will amend it as required post a discussion of spherical symmetry consideration.

Theorem 6 (Christodolou, 1983)**.** *For the spherically symmetric Einstein dust system, generic and arbitrarily small perturbations of the homogeneous data on Σ give rise to a maximal Cauchy development $(\mathcal{M}, g)$ that is smoothly extendible across a Cauchy horizon $\mathcal{CH}^+$. [Chr84]*

This theorem corresponds to Figure 8.2. We see that $\mathcal{I}^+$ is complete. Contrast this with Figure 8.3, where a Cauchy horizon is also present but $\mathcal{I}^+$ is incomplete. In both spacetimes, the manifold is extendible beyond the Cauchy horizon. The first Penrose diagram in Figure 8 is consistent with every formulation of cosmic censorship we have stated thus far but by Christodolou's theorem, weak and strong cosmic censorship fail for the Einstein dust system because as demonstrated in [Chr84]. However, we know that dust is not a very good model for matter, so we can examine the Einstein scalar field system instead to get a better idea of the validity of the conjectures in more astrophysically accurate scenarios.

Theorem 7 (Christodolou, 1990)**.** *For the spherically symmetric Einstein scalar field system, there exist regular, complete, asymptotically flat initial data on Σ giving rise to a maximal Cauchy development $(\mathcal{M}, g)$ with the Penrose diagrams Figures 8.2, 8.3.*

By this theorem, we see that our initial attempt at formulating weak cosmic censorship in a modern fashion fails. Fortunately, there is a simple fix – we introduce the genericity condition for the initial data.

Conjecture 4 (Weak cosmic censorship, amended)**.** *For generic and asymptotically flat vacuum initial data, the maximal Cauchy development has a complete null infinity $\mathcal{I}^+$.*

The spherically symmetric Einstein dust and scalar field systems mentioned in these theorems do not have this genericity condition, hence we can no longer consider the issue of cosmic censorship using these examples. With this genericity condition taken on the spherically symmetric Einstein scalar field system, Christodolou formulated the following:

Theorem 8 (Christodolou, 1999)**.** *Weak, strong, and very strong cosmic censorship hold for the spherically symmetric Einstein scalar field system, i.e. for generic spherically symmetric initial data, the maximal future Cauchy development has Penrose diagram Figure 8.1. with complete future null infinity and a spacelike singularity.*

Hence, we may say that given the initial data satisfying genericity, [Chr84] shows that the scalar field satisfies our mathematical criteria to a sufficient extent to be considered representative of suitable matter.

4 Conclusion

The cosmic censorship conjectures arose from the foundations laid by Hawking and Penrose via their singularity theorems. The original formulations of the conjectures were based on the idea of unstoppable collapse resulting in the creation of a trapped surface, as illustrated by the classic example of the coordinate and curvature singularities inherent in the Schwarzschild metric. These formulations were rooted in the machinery of causality. However, the conjectures were actually deeper statements about global existence and uniqueness, accessible via the machinery of Cauchy evolution. The modern formulations are rooted in the assumption of the primacy of the Cauchy problem as posed by Cauchy-Bruhat, which ultimately provide GR with its deterministic character as a physical theory.

References

[OS39] J Robert Oppenheimer and Hartland Snyder. “On continued gravitational contraction”. In: *Physical Review* 56.5 (1939), p. 455.

[Pen65] Roger Penrose. “Gravitational collapse and space-time singularities”. In: *Physical Review Letters* 14.3 (1965), p. 57.

[CG69] Yvonne Choquet-Bruhat and Robert Geroch. “Global aspects of the Cauchy problem in general relativity”. In: *Communications in Mathematical Physics* 14 (1969), pp. 329–335.

[Pen69] R Penrose. “Gravitational collapse: The role of general relativity”. In: (1969).

[HP70] Stephen William Hawking and Roger Penrose. “The singularities of gravitational collapse and cosmology”. In: *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 314.1519 (1970), pp. 529–548.

[GH79] Robert Geroch and Gary T Horowitz. “Global structure of spacetimes”. In: *General relativity.* 1979.

[Chr84] Demetrios Christodoulou. “Gravitational Collapse of a Dust Cloud and the Cosmic Censorship Conjecture”. In: *General Relativity and Gravitation: Invited Papers and Discussion Reports of the 10th International Conference on General Relativity and Gravitation, Padua, July 3–8, 1983.* Springer. 1984, pp. 27–36.

[Chr99] Demetrios Christodoulou. “On the global initial value problem and the issue of singularities”. In: *Classical and Quantum Gravity* 16.12A (1999), A23.

[Wal10] Robert M Wald. *General relativity.* University of Chicago press, 2010.

[Sbi15] Jan Sbierski. “Characterisation of the energy of Gaussian beams on Lorentzian manifolds: with applications to black hole spacetimes”. In: *Analysis & PDE* 8.6 (2015), pp. 1379–1420.

[Car19] Sean M Carroll. *Spacetime and geometry.* Cambridge University Press, 2019.

[HE23] Stephen W Hawking and George FR Ellis. *The large scale structure of space-time.* Cambridge university press, 2023.